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Elena Chierici

The cone of curves of Fano varieties

Relatori

Prof. Marco Andreatta

Dott. Gianluca Occhetta

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To Luca, Giovanni and Chiara

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Contents

Introduction

Mori's Minimal Model Program is undoubtly the most significant step of the last decades towards the birational classification of algebraic varieties: while the case of curves was already settled in the XIX century and the italian school of Castelnuovo, Enriques and Severi had completed the classification of surfaces at the beginning of the XX, higher dimensional varieties still lacked a powerful tool to be investigated with. The classification of algebraic surfaces had shown how birational transformations could be employed to find a "model" in any birational class, which was called a minimal surface. Similarly, the aim of MMP as introduced by Mori at the beginning of the 1980s is to find "simple" objects inside a birational class of varieties, so that the knowledge of the minimal model of a variety X together with the birational morphisms from X to this model can disclose some properties of the the variety itself.

As in the case of surfaces, the first question one has to pose is whether the canonical divisor K_X is nef or not, i.e. if there exists a curve on X such that $K_X \cdot C < 0$. If K_X is nef we call X a minimal model, and it is conjectured that in this case X has a unique canonical model with only canonical singularities or is an algebraic fiber space onto a normal projective variety with rational singularities; otherwise we can find some particular morphisms which are defined on X and which provide some relevant information about its structure. Those are the so-called contractions, i.e. proper morphisms $f : X \to Y$ with connected fibers between normal varieties. A way to describe these morphisms is to look at the curves they contract; this can be done considering in the $\mathbb{R}$-vector space $N_1(X)$ of 1-cycles up to numerical equivalence, whose (finite) dimension is usually denoted by ρ_X and called the Picard number of X, the convex cone $\mathrm{NE}(X)$ generated by the classes of irreducible

effective curves.
It is an easy remark that the classes of curves contracted by a given morphism generate a convex extremal subcone of $\mathrm{NE}(X)$; on the contrary, it's a highly non-trivial problem to decide whether an extremal subcone of $\mathrm{NE}(X)$ corresponds to a contraction.
A partial but significant answer to this problem comes from Mori theory: first of all the Cone theorem provides a description of the structure of the part of $\mathrm{NE}(X)$ which lies in the semispace of $N_1(X)$ where the intersection with the canonical bundle K_X is negative; then the Contraction theorem ensures that to every extremal face in this subcone one can associate a contraction. In general we have no information about the "positive" part of the cone, so Mori theory is a powerful tool to study varieties whose canonical bundle is not nef.
In particular the Cone and Contraction theorems completely describe the subcone associated to the contractions of those varieties whose cone of curves is entirely contained in the semispace where $K_X < 0$; by Kleiman's criterion this condition is equivalent to the ampleness of the anticanonical bundle $-K_X$, and varieties with this property are called Fano varieties.

To a smooth Fano variety X one can associate:

r_X, the index of X, which is the largest natural number m such that $-K_X = mH$ for some (ample) divisor H on X;

i_X, the pseudoindex of X, which is defined as

$$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m \text{ for some rational curve } C \subset X\}.$$

It is known that $0 < r_X \leq \dim X + 1$, and a theorem of Kobayashi and Ochiai [KO73] gives a complete characterization of Fano varieties with $r_X \geq \dim X$. Moreover, the classification of del Pezzo surfaces and Fano threefolds and the application of the Apollonius method, which consists in reducing to lower-dimensional varieties studying hyperplane sections of X, have led to the classification of Fano varieties of index $\dim X - 1$, which are called del Pezzo varieties, and of index $\dim X - 2$, called Mukai varieties (see [AM03] for a review).

Since the classification of Fano fourfolds is very far from being known, it is not possible to use Apollonius method to study Fano varieties of index $\dim X - 3$; however, according to what we said up to now, a first step towards the classification of these varieties could be the description of their cone of curves, namely of its dimension and of the number and type of the extremal rays which generate it.

Concerning the first problem, i.e. the dimension of $\mathrm{NE}(X)$, a bound was conjectured by Mukai in [Muk88]:

Conjecture A. *If X is a Fano variety of dimension n, then*

$$\rho_X(r_X - 1) \leq n.$$

Since it is clear from the definitions that $r_X \leq i_X$, conjecture A can be generalized as follows:

Conjecture B. *If X is a Fano variety of dimension n, then*

$$\rho_X(i_X - 1) \leq n,$$

and equality holds if and only if $X \simeq (\mathbb{P}^{i_X-1})^{\rho_X}$.

A well-known theorem due to Mori states that $i_X \leq n+1$ [Mor79]; in 1990 Wiśniewski [Wiś90b] proved that if $i_X > \frac{n+2}{2}$ then $\rho_X = 1$, and in that paper he implicitly noticed that the statement of conjecture B is more natural. In 2002 Bonavero, Casagrande, Debarre and Druel [BCDD03] explicitely posed conjecture B and proved it in the following situations:

(a) X has dimension 4,
(b) X is a toric variety of pseudoindex $i_X \geq \frac{n+3}{3}$,
(c) X is a toric variety of dimension ≤ 7,
(d) X is a homogeneous Fano variety.

In the first part of this thesis we will give the proof of the following theorems (see [ACO04]):

Theorem 1. *Let X be a Fano variety of dimension n and pseudoindex $i_X \geq \frac{n+3}{3}$; then conjecture B holds if X has a covering unsplit family of rational curves.*

Theorem 2. *If X is a Fano fivefold then conjecture B holds for X.*

The proof of these results is deeply based on Mori's work: the first step towards the Cone theorem was the proof that through every point of a variety X which lies on a curve where K_X is not nef there passes a rational curve whose anticanonical degree is $\leq n+1$; in particular this implies that Fano varieties are covered by

rational curves.
Moreover, rational curves can be arranged in families, i.e. they can be parametrized by suitable subschemes of the Chow variety of X.
One can use these families to define relations on the points of X; e.g. two points x and y can be considered equivalent if they can be connected by a chain of rational curves belonging to some chosen families. If these families are proper (i.e. the corresponding component of $\mathrm{Chow}_1(X)$ is proper) then there exists a proper fibration, defined on an open subset of X, whose general fiber is an equivalence class; this map has the interesting property that one can write the numerical class of every curve contained in a general fiber as a linear combination of the classes of the irreducible components of the 1-cycles parametrized by the families involved in the fibration.
A remarkable result on Fano varieties is that if we consider the fibration associated to all the components of $\mathrm{Chow}_1(X)$ parametrizing rational 1-cycles, then the equivalence class is the whole X, i.e. the variety is rationally connected ([KMM92]). However, in general the rational connectedness of X can be obtained using only some components of $\mathrm{Chow}_1(X)$; so a strategy to bound the Picard number of X is to find the smallest number of families which do the job and then bound the number of numerically independent components of the 1-cycles they parametrize.

Concerning the second problem, i.e. the description of the number and type of the extremal rays of a Fano variety X, we will prove the following theorems (see [CO04]):

Theorem 3. *Let X be a Fano variety of dimension $n \geq 5$, pseudoindex $i_X = n-3$ and Picard number $\rho_X \geq 2$. Then $\mathrm{NE}(X)$ is generated by ρ_X rays.*
More precisely, we have the following list of possibilities, where F stands for a fiber type contraction, D_i for a divisorial contraction whose exceptional locus is mapped to a $i-$dimensional subvariety and S for a small contraction. All cases are effective.

$\dim X$	ρ_X	R_1	R_2	R_3	R_4	R_5
5	2	F	F			
		F	D_0			
		F	D_1			
		F	D_2			
		F	S			
		D_2	D_2			
		D_2	S			
	3	F	F	F		
		F	F	S		
		F	F	D_1		
		F	F	D_2		
		F	D_2	D_2		
	4	F	F	F	F	
		F	F	F	D_2	
	5	F	F	F	F	F
6	2	F	F			
		F	D_1			
		F	D_2			
		F	S			
	3	F	F	F		
7	2	F	F			
		F	D_2			
8	2	F	F			

Theorem 4. *Let X be a Fano fivefold of pseudoindex $i_X = 2$ which does not have a covering quasi-unsplit locally unsplit family of rational curves; then $\rho_X = 2$, and X is the blow-up of $\mathbb{P}^5$ along a two-dimensional smooth quadric, or along a cubic scroll $\subset \mathbb{P}^4$, or along a Veronese surface.*

The thesis is organized as follows:
chapters 1 to 3 are dedicated to the theory of rational curves on projective varieties; in particular we recall the construction of their parametrizing schemes (chapter 1), the basic results of Mori theory (chapter 2), and how rational curves can be organized in families (chapter 3).

In chapter 4 we introduce the object of the thesis, Fano varieties.
In chapter 5 we give some upper bounds on their Picard number, and we prove Mukai conjecture for special Fano varieties; we conclude this chapter with the proof of theorem 3 for Fano varieties of dimension ≥ 6.
Chapters 6 and 7 are devoted to the study of Fano fivefolds, while chapter 8 contains all the examples of the cases listed in theorem 3.

We work over the field $\mathbb{C}$ of complex numbers, and our notation is consistent with the usual one, as for instance in [Har77], [Kol96] and [Deb01].

1

Rational curves

For all the material in this chapter the main references are the first chapters of [Kol96] and [Deb01].

1.1 Preliminaries

Let X be a proper scheme of dimension n and let $D_1, \ldots, D_r$ be Cartier divisors on X. Assume that $r \geq n$.

Definition 1.1.1. The intersection number $D_1 \cdot \ldots \cdot D_r$ is the coefficient of $m_1 \cdots m_r$ in the polynomial

$$\chi(X, m_1 D_1 + \cdots + m_r D_r) := \sum_i (-1)^i h^i(X, m_1 D_1 + \cdots + m_r D_r).$$

If Y is a closed subscheme of X of dimension $\leq s$ we also set

$$D_1 \cdot \ldots \cdot D_s \cdot Y = D_{1|Y} \cdot \ldots \cdot D_{s|Y}.$$

Remark 1.1.2. If $r > n$ then $D_1 \cdot \ldots \cdot D_r = 0$.

If D is a Cartier divisor and C is a complete curve on X (i.e. an integral proper one-dimensional subscheme of X), we can consider the intersection number $D \cdot C$, which is by definition 1.1.1 the leading coefficient of the polynomial $\chi(X, mD_{|C})$. From Riemann-Roch theorem (see [Har77, IV.1.3]) we know that

$$\chi(X, mD_{|C}) = m \deg(\mathcal{O}_C(D)) + \chi(C, \mathcal{O}_C),$$

hence

$$D \cdot C = \deg(\mathcal{O}_C(D)).$$

This last definition can be extended by linearity to the group of formal linear combinations of curves with integral coefficients. Such a linear combination $\Gamma = \sum a_i C_i$ is called a 1-cycle on X, and if all the coefficients are nonnegative the 1-cycle Γ is called effective.

Notation. Let X be a proper scheme; we denote by $\mathrm{Div}(X)$ the group of Cartier divisors on X and by $Z_1(X)$ the free abelian group generated by the 1-cycles on X.

Let $f : X \to Y$ be a proper morphism, let Γ an irreducible 1-cycle on X and set $\Gamma' := f(\Gamma)$. Then we can define the push-forward $f_* : Z_1(X) \to Z_1(Y)$ as

$$f_*\Gamma = \begin{cases} \deg(f_{|\Gamma})\Gamma' & \text{if } \dim \Gamma' = 1 \\ 0 & \text{if } \dim \Gamma' = 0 \end{cases}$$

Definition 1.1.3. Let S be a normal surface and X a proper scheme. We say that two effective 1-cycles Δ, $\Delta' \in Z_1(S)$ are effectively algebraically equivalent if there exist a proper flat morphism $p : S \to C$ onto a smooth curve C and two points x, $x' \in C$ such that $\Delta = p^{-1}(x)$ and $\Delta' = p^{-1}(x')$.
We say that two effective 1-cycles Γ, $\Gamma' \in Z_1(X)$ are effectively algebraically equivalent if there exist a normal surface S, a proper morphism $g : S \to X$ and two effectively algebraically equivalent 1-cycles Δ, $\Delta' \in Z_1(S)$ such that $\Gamma = g_*\Delta$ and $\Gamma' = g_*\Delta'$.
The transitive hull of this relation defines an equivalence relation on $Z_1(X)$, which we call effective algebraic equivalence.

Theorem 1.1.4. *There exists a projective scheme* $\mathrm{Chow}_1(X)$*, which parametrizes effective 1-cycles on* X*, with the property that if two 1-cycles belong to the same irreducible component of* $\mathrm{Chow}_1(X)$ *then they are effectively algebraically equivalent.*

Definition 1.1.5. We say that two 1-cycles Γ, $\Gamma' \in Z_1(X)$ are algebraically equivalent if there exists a 1-cycle E such that $\Gamma + E$ and $\Gamma' + E$ are effectively algebraically equivalent.

Definition 1.1.6. We say that two Cartier divisors D, $D' \in \mathrm{Div}(X)$ are numerically equivalent, and we write $D \equiv D'$, if

$$D \cdot C = D' \cdot C$$

for every curve $C \subset X$. The quotient of $\mathrm{Div}(X)$ by this equivalence relation is denoted by $\mathrm{N}^1(X)_{\mathbb{Z}}$, and we can also consider the $\mathbb{R}$-vector space

$$\mathrm{N}^1(X) := \mathrm{N}^1(X)_{\mathbb{Z}} \otimes \mathbb{R}.$$

Definition 1.1.7. We say that two 1-cycles Γ, $\Gamma' \in Z_1(X)$ are numerically equivalent if

$$D \cdot \Gamma = D \cdot \Gamma'$$

for every Cartier divisor $D \in \mathrm{Div}(X)$. The quotient of $Z_1(X)$ by this equivalence relation is denoted by $\mathrm{N}_1(X)_{\mathbb{Z}}$, and we can also consider the $\mathbb{R}$-vector space

$$\mathrm{N}_1(X) := \mathrm{N}_1(X)_{\mathbb{Z}} \otimes \mathbb{R}.$$

If no confusion can arise, we will denote by Γ both a 1-cycle in $Z_1(X)$ and its numerical equivalence class in $\mathrm{N}_1(X)$.

Remark 1.1.8. Note that if Γ, $\Gamma' \in Z_1(X)$ are algebraically equivalent then they are also numerically equivalent.

The intersection form induces a nondegenerate pairing

$$\mathrm{N}^1(X) \times \mathrm{N}_1(X) \longrightarrow \mathbb{R}$$

which makes these vector spaces canonically dual. Moreover, they are finite-dimensional by the Néron-Severi theorem, and the number

$$\rho_X = \dim \mathrm{N}^1(X) = \dim \mathrm{N}_1(X)$$

is called the Picard number of X.

Definition 1.1.9. We define the Mori cone $\mathrm{NE}(X) \subset \mathrm{N}_1(X)$ as the convex cone generated by the effective 1-cycles on X.

Notation. If $D \in \mathrm{N}^1(X)$ is a Cartier divisor we write

$$\mathrm{NE}(X)_{D\geq 0} = \{\Gamma \in \mathrm{NE}(X) \mid D \cdot \Gamma \geq 0\},$$

and similarly for $\mathrm{NE}(X)_{D\leq 0}$, $\mathrm{NE}(X)_{D>0}$, $\mathrm{NE}(X)_{D<0}$, $\mathrm{N}_1(X)_{D\geq 0}$, etc.

Definition 1.1.10. Let X be a proper scheme and D a Cartier divisor on X. We say that D is ample if some multiple of D is very ample, i.e. the associated map $\varphi_{|D|} : X \to \mathbb{P}^{h^0(X,\mathcal{O}(D))-1}$ is an embedding. The variety X is said to be projective if on X there exists an ample divisor.

For a projective variety we have the following numerical characterization of ampleness:

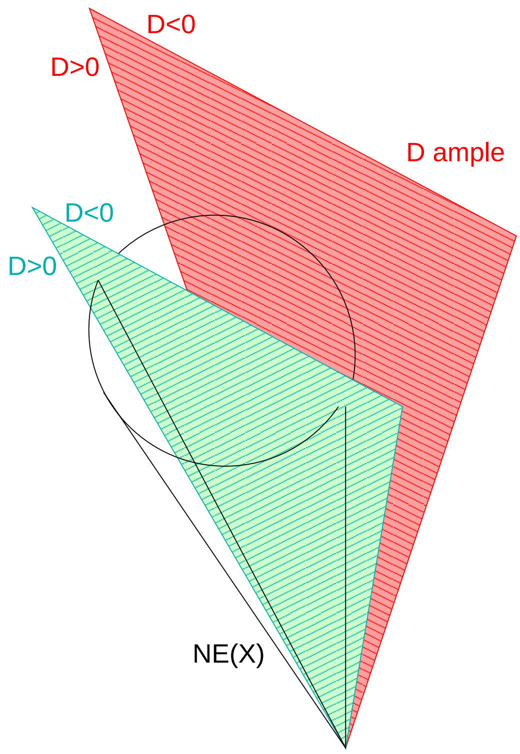
Theorem 1.1.11 (Kleiman's criterion). *A Cartier divisor D on a projective variety X is ample if and only if*

$$D \cdot \Gamma > 0 \quad \textit{for every} \quad \Gamma \in \overline{\mathrm{NE}(X)} \setminus \{0\}.$$

Moreover, for every ample divisor D on X and for every integer k, the set

$$\{\Gamma \in \mathrm{NE}(X) \mid D \cdot \Gamma \leq k\}$$

is compact, hence it contains a finite number of numerical classes of irreducible curves.



This characterization leads to the following natural extension of the ampleness property:

Definition 1.1.12. Let X be a proper scheme and D a Cartier divisor on X. We say that D is numerically effective, or nef, if

$$D \cdot \Gamma \geq 0 \quad \text{for every} \quad \Gamma \in \overline{\mathrm{NE}(X)} \setminus \{0\},$$

or equivalently

$$D \cdot C \geq 0 \quad \text{for every curve} \quad C \subseteq X.$$

Kleiman's criterion shows that ampleness is a numerical property, and so is nefness, so we can talk about ample and nef classes of Cartier divisors in $\mathrm{N}^1(X)$; moreover it follows easily from Kleiman's criterion that the ample classes generate an open cone in $\mathrm{N}^1(X)$, which is called the ample cone and whose closure coincides with the nef cone, i.e. the cone generated by the classes of nef divisors on X.

Another immediate consequence of Kleiman's criterion concerns the structure of the Mori cone of a projective variety:

Corollary 1.1.13. *The Mori cone* $\mathrm{NE}(X)$ *of a projective variety* X *contains no lines, i.e. it is entirely contained in an open half-space of* $\mathrm{N}_1(X)$ *plus the origin.*

1.2 Parametrizing schemes

Let Y be a projective variety and let X be a smooth quasi-projective variety; morphisms $f : Y \to X$ can be parametrized by a locally Noetherian scheme $\mathrm{Hom}(Y, X)$, whose points will be denoted by $[f]$ and which has the following universal property: for every scheme D and for every morphism $F : Y \times D \to X$ there exists a unique morphism $F' : D \to \mathrm{Hom}(Y, X)$ such that the diagram

$$\begin{tikzcd} Y \times \mathrm{Hom}(Y,X) \arrow[r, "e"] & X \\ Y \times D \arrow[u, "Id \times F'"] \arrow[ur, "F"'] & \end{tikzcd}$$

commutes, where $e : Y \times \mathrm{Hom}(Y, X) \to X$ denotes the evaluation map which sends $(y, [f])$ to $f(y)$.

In general the scheme $\mathrm{Hom}(Y, X)$ has countably many components, but each irreducible component is in fact a quasi-projective variety. The following theorem (see [Kol96, II.1.7]) provides very important informations about its local structure:

Theorem 1.2.1. *Let $f_0 : Y \to X$ be a morphism from a projective variety Y to a smooth quasi-projective variety X. Then*

(a) the Zariski tangent space to $\mathrm{Hom}(Y, X)$ *is*

$$T_{[f_0]} \mathrm{Hom}(Y, X) \simeq H^0(Y, f_0^* TX),$$

where TX denotes the tangent bundle of X;

(b) $\dim_{[f_0]} \mathrm{Hom}(Y, X) \geq h^0(Y, f_0^* TX) - h^1(Y, f_0^* TX)$*;*

(c) if $H^1(Y, f_0^* TX) = 0$ then $\mathrm{Hom}(Y, X)$ *is smooth at $[f_0]$ and has dimension $h^0(Y, f_0^* TX)$.*

The same construction holds if we consider morphisms from Y to X which fix a closed subscheme $B \subset Y$; more precisely, if $g : B \to X$ is a given morphism we can consider the scheme $\mathrm{Hom}(Y, X; g)$ which parametrizes morphisms $f : Y \to X$ such that $f_{|B} = g$.

Clearly $\mathrm{Hom}(Y, X; g)$ is a subscheme of $\mathrm{Hom}(Y, X)$, and it has the same properties; moreover theorem 1.2.1 still holds up to replace the sheaf $f_0^* TX$ with $f_0^* TX \otimes \mathcal{I}_B$, where $\mathcal{I}_B$ denotes the ideal sheaf of B in Y.

1.3 Parametrizing curves on varieties

A particular case of what we said up to now is the scheme which parametrizes curves on a variety X, i.e. morphisms $f : C \to X$ where C is a proper curve without embedded points and X is a smooth quasi-projective variety. In this case the previous theorems get simpler:

Theorem 1.3.1. *Let X be a smooth quasi-projective variety, C a proper curve without embedded points of genus $g(C)$, and $f : C \to X$ a morphism. Then*

(a) $T_{[f]} \operatorname{Hom}(C, X) \simeq H^0(C, f^*TX)$;
(b) $\dim_{[f]} \operatorname{Hom}(C, X) \geq -K_X \cdot f_*C + \dim X(1 - g(C))$.

Theorem 1.3.2. *Let X be a smooth quasi-projective variety, C a proper curve without embedded points of genus $g(C)$, and $f : C \to X$ a morphism. Let B be a closed subscheme of C of finite length $l(B)$ and $g : B \to X$ a morphism. Then*

(a) $T_{[f]} \operatorname{Hom}(C, X; g) \simeq H^0(C, f^*TX \otimes \mathcal{I}_B)$;
(b) $\dim_{[f]} \operatorname{Hom}(C, X; g) \geq -K_X \cdot f_*C + \dim X(1 - g(C) - l(B))$.

1.4 Parametrizing rational curves

Let X be a normal projective variety and let $\operatorname{Hom}(\mathbb{P}^1, X)$ be the scheme parametrizing morphisms $f : \mathbb{P}^1 \to X$; we consider $\operatorname{Hom}_{bir}(\mathbb{P}^1, X) \subset \operatorname{Hom}(\mathbb{P}^1, X)$, the open subscheme corresponding to those morphisms which are birational onto their image, and its normalization $\operatorname{Hom}^n_{bir}(\mathbb{P}^1, X)$; since every nonconstant morphism $f : \mathbb{P}^1 \to X$ factors through a map $\mathbb{P}^1 \to \mathbb{P}^1$ and a morphism $g : \mathbb{P}^1 \to X$ which is birational onto its image, all informations about $\operatorname{Hom}(\mathbb{P}^1, X)$ are in fact contained in $\operatorname{Hom}^n_{bir}(\mathbb{P}^1, X)$, at least set-theoretically. Nevertheless, if h is any automorphism of $\mathbb{P}^1$ and $f \in \operatorname{Hom}^n_{bir}(\mathbb{P}^1, X)$, then $f \circ h$ is counted as a different morphism, while for our purposes they should be considered as the same rational curve. For this reason we consider the group action of $\operatorname{Aut}(\mathbb{P}^1)$ on $\operatorname{Hom}^n_{bir}(\mathbb{P}^1, X)$ and we focus our attention on the quotient:

Definition 1.4.1. The space $\operatorname{Ratcurves}^n(X)$ is the quotient of $\operatorname{Hom}^n_{bir}(\mathbb{P}^1, X)$ by $\operatorname{Aut}(\mathbb{P}^1)$, and the space $\operatorname{Univ}(X)$ is the quotient of the product action of $\operatorname{Aut}(\mathbb{P}^1)$ on the space $\operatorname{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1$.

We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \arrow[r, "U"] \arrow[d] & \mathrm{Univ}(X) \arrow[d, "p"] \\
\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \arrow[r, "u"] & \mathrm{Ratcurves}^n(X)
\end{tikzcd} \tag{1.1}$$

where u and U are principal $\mathrm{Aut}(\mathbb{P}^1)$-bundles and p is a $\mathbb{P}^1$-bundle.

If we fix a point $x \in X$, this construction can be repeated starting from the scheme $\mathrm{Hom}(\mathbb{P}^1, X; 0 \mapsto x)$ which parametrizes morphisms $f : \mathbb{P}^1 \to X$ sending $0 \in \mathbb{P}^1$ to x. Again we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X; 0 \mapsto x) \times \mathbb{P}^1 \arrow[r, "U"] \arrow[d] & \mathrm{Univ}(X, x) \arrow[d, "p"] \\
\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X; 0 \mapsto x) \arrow[r, "u"] & \mathrm{Ratcurves}^n(X, x)
\end{tikzcd}$$

Remark 1.4.2. For every integer $d \geq 0$ we can consider the quasi-projective scheme $\mathrm{Hom}_d(\mathbb{P}^1, X)$ which parametrizes morphisms $\mathbb{P}^1 \to X$ of degree d with respect to a given ample divisor, and the space $\mathrm{Hom}(\mathbb{P}^1, X)$ can be written as the disjoint union

$$\bigcup_{d \geq 0} \mathrm{Hom}_d(\mathbb{P}^1, X).$$

This implies that on a projective variety X there exist only countably many numerical classes of rational curves. Moreover, for every positive integer d and any ample divisor H there exists only a finite number of numerical classes of rational curves of H-degree $\leq d$.

2

Mori theory for smooth varieties

2.1 The bend & break technique

The “bend & break” technique was first introduced by Mori in his famous paper [Mor79], where he proved Hartshorne’s conjecture about varieties with ample tangent bundle. However, his techniques have turned out to be a very powerful tool for investigating the birational geometry of algebraic varieties, and in particular they fit very well to the study of Fano varieties, as we will see in the next sections.

Mori’s main idea is the following: if a curve on a variety X deforms nontrivially while keeping a point fixed, at some point it breaks up into an effective 1-cycle with a rational component passing through the fixed point. This allows to find rational curves on a variety starting from a curve of any genus, provided that its space of deformations is sufficiently “big” (proposition 2.1.1); moreover, if a rational curve deforms nontrivially while keeping two points fixed, it must break up into an effective reducible 1-cycle with rational components (proposition 2.1.2). In the next section we will use these lemmas to show the existence of rational curves on varieties whose canonical bundle is not nef.

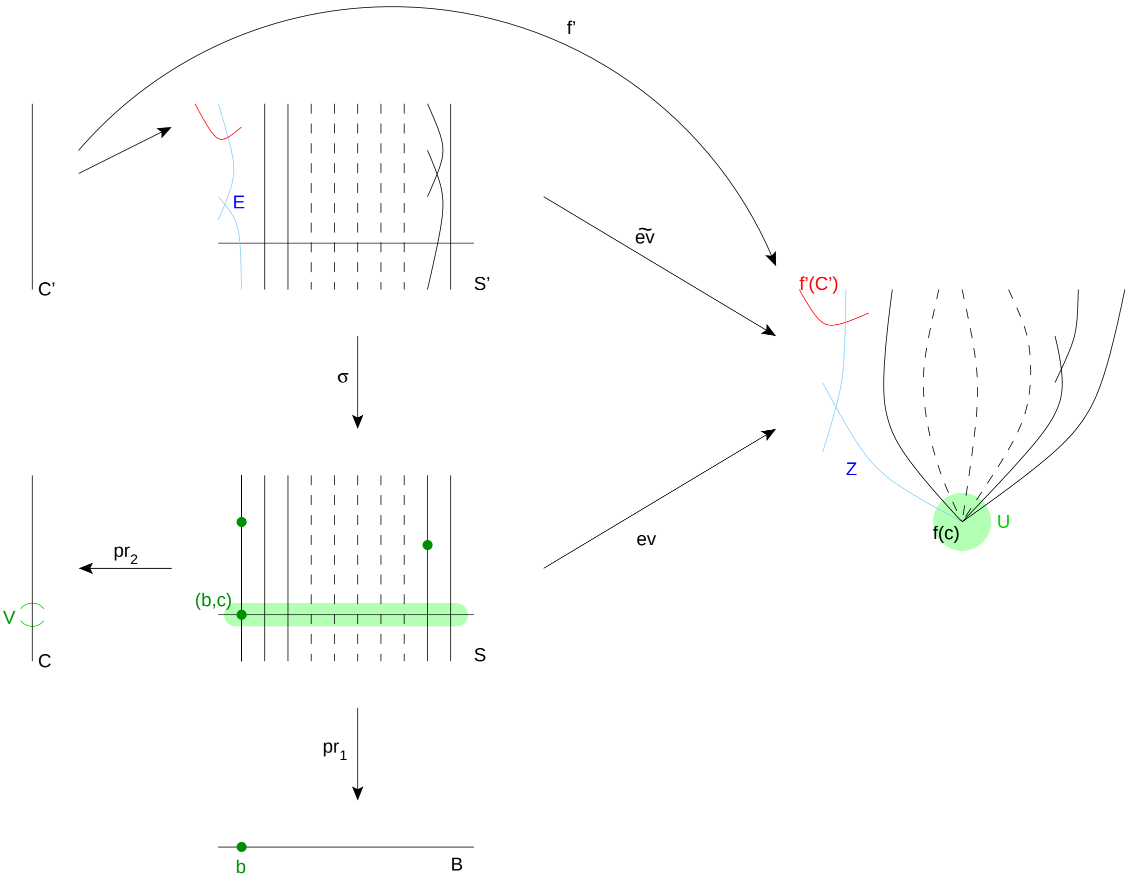
Lemma 2.1.1 (Bend & break I). *Let $f : C \to X$ be a smooth curve on a projective variety X, and let c be a point on C. If*

$$\dim_{[f]} \operatorname{Hom}(C, X; f_{|\{c\}}) \geq 1$$

then there exist a curve $f' : C' \to X$ and a connected effective nonzero rational 1-cycle Z on X which passes through $f(c)$ and satisfies

$$f_*C \sim f'_*C' + Z.$$

Proof. We may assume that C is not rational, otherwise the statement is trivial. Consider a 1-dimensional subvariety of $\mathrm{Hom}(C, X; f_{|\{c\}})$ which contains $[f]$, call B_0 its normalization and consider a smooth compactification B of B_0; let $S := B \times C$, let $ev : B \times C \dashrightarrow X$ be the rational map induced by the evaluation and let $\sigma : S' \to S$ be a resolution of the indeterminacies of ev.



First of all we show that there exists a point $b \in B$ such that ev is not defined at the point (b, c). Suppose by contradiction that ev is defined at every point of $B \times \{c\}$, consider in X an open affine neighbourhood U of $f(c)$ and choose in C an open neighbourhood V of c such that $ev(B \times V) \subseteq U$. For every $v \in V$ the image of $B \times V$ is a complete subvariety of U, hence it is a point; it follows that ev has infinitely many 1-dimensional fibers, so its image must coincide with the curve $f(C)$. On the other hand, since the subgroup of $\mathrm{Aut}(C)$ which fixes the point c is finite, we have that $\dim ev(B \times C) = 2$, a contradiction.

The fiber of $pr_1 \circ \sigma$ over the point b consists of the strict transform of C and a connected rational 1-cycle E such that E is not contracted by $\widetilde{ev}$ and $(b, c) \in \sigma(E)$; hence the rational 1-cycle $Z = \widetilde{ev}(E)$ passes through $ev(b, c) = f(c)$. $\square$

Lemma 2.1.2 (Bend & break II). *Let X be a projective variety and let $f : \mathbb{P}^1 \to X$ be a rational curve. If*

$$\dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, X; f_{|\{0,\infty\}}) \geq 2$$

then the 1-cycle $f_*\mathbb{P}^1$ is numerically equivalent to a connected nonintegral effective rational 1-cycle Z passing through $f(0)$ and $f(\infty)$.

Here, the difference with respect to the proof of the previous proposition is that a new situation can arise, where the image via $\widetilde{ev}$ of every reducible fiber of $pr_1 \circ \sigma$ is irreducible (i.e. the strict transforms of all fibers of pr_1 which contain some indeterminacy point of ev are contracted by $\widetilde{ev}$).
In this case $\widetilde{ev}$ factors through a normal surface $\widetilde{S}$ which has a fibration over B; if all fibers of this fibration were irreducible, then $\widetilde{S}$ would be a $\mathbb{P}^1$-bundle (see for instance [Kol96, II.2.8]), and it could not contain two sections which are both contracted and so by [Kol96, II.5.3.2] have negative self-intersection.

2.2 Existence of rational curves

Theorem 2.2.1. *[Mor79] Let X be a smooth projective variety of dimension n such that $-K_X$ is ample. Then through any point of X there exists a rational curve $\mathit{\Gamma} \subset X$ satisfying $-K_X \cdot \mathit{\Gamma} \leq n + 1$.*

Proof. We divide the proof into steps:

Step 1. *Reduction to characteristic p.*

Let $C \subset X$ be a curve of positive genus; denote with $f_1, \dots, f_k$ the polynomials which define X and with $g_1, \dots, g_j$ the polynomials which define C in $\mathbb{P}^N$, and assume that they all have integral coefficients.
For a fixed prime p, let $\mathbb{F}_p$ be the finite field of the integers mod p and denote by $\overline{\mathbb{F}}_p$ its algebraic closure; then the above equations define subvarieties X_p, C_p of $\mathbb{P}^N_{\overline{\mathbb{F}}_p}$, which are nonsingular.
Now consider the Frobenius morphism

$$\begin{aligned} F_p : \mathbb{P}^N_{\overline{\mathbb{F}}_p} &\longrightarrow \mathbb{P}^N_{\overline{\mathbb{F}}_p} \\ (x_0 : \ldots : x_N) &\longmapsto (x_0^p : \ldots : x_N^p); \end{aligned}$$

if f is a polynomial in $\mathbb{F}_p[x_0, \ldots, x_N]$ we have that

$$f \circ F_p(x_0, \ldots, x_N) = f(x_0^p, \ldots, x_N^p) = f(x_0, \ldots, x_N)^p,$$

i.e. the Frobenius morphism maps C_p into itself. Iterating it m times and composing with the inclusion $C_p \to X_p$ we obtain a morphism

$$F := F_p^m : C_p \longrightarrow X_p$$

satisfying

$$\dim_{[F]} \operatorname{Hom}(C_p, X_p; F_{|\{c\}}) \geq p^m(-K_{X_p} \cdot C_p) - g(C_p) \dim X.$$

By "generic flatness over Spec$\mathbb{Z}$", the integers $-K_{X_p} \cdot C_p$ and $g(C_p)$ are constant for almost every p, hence it is possible to pick a sufficiently large m such that the above expression is positive for almost every p.

Now we apply the following principle: if a homogeneous system of polynomial equations with integral coefficients has a nontrivial solution in $\overline{\mathbb{F}}_p$ for infinitely many p, then it has a nontrivial solution in every algebraically closed field. Then we can apply theorem 2.1.1 and find a rational curve $f' : \mathbb{P}^1 \to X_p$; if $-K_X \cdot f'(\mathbb{P}^1) \leq n+1$ we conclude, otherwise $\dim_{[f']} \operatorname{Hom}(\mathbb{P}^1, X_p; f'_{|\{0,\infty\}}) \geq 2$ and we can apply theorem 2.1.2. So the theorem is proved over a field of positive characteristic.

Step 2. *From characteristic p to characteristic zero.*

Let X be defined over a field $\mathbb{K}$ of characteristic zero, and let R be the subring of $\mathbb{K}$ generated by the coefficients of $f_1, \ldots, f_k$ and by the coordinates of a point $x = (x_0 : \ldots : x_N)$ (we can assume $\mathbb{K} = \mathbb{C}$ and $R = \mathbb{Z}$).

The equations $f_1 = \cdots = f_k = 0$ define in $\mathbb{P}^N_R$ a scheme $\mathcal{X}$ which has a point of coordinates $(x_0 : \ldots : x_N)$, and the generic fiber of $\mathcal{X} \to \operatorname{Spec}(R)$ corresponds to the quotient field $K(R)$ of R; moreover for base change we obtain X, as shown in the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_{K(R)} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec}(\mathbb{K}) \arrow[r] & \operatorname{Spec}(K(R)) \arrow[r] & \operatorname{Spec}(R) \end{tikzcd}$$

Lemma 2.2.2. *For every maximal ideal $\mathfrak{m} \subset R$ the field $R/\mathfrak{m}$ is finite; moreover maximal ideals are dense in* $\operatorname{Spec}(R)$.

On $\operatorname{Spec}(R)$ there exists a quasi-projective scheme

$$\mathcal{H} = \bigcup_{0 \le d \le n+1} \operatorname{Hom}_d(\mathbb{P}^1, \mathcal{X}; 0 \longmapsto x_R),$$

where d is the degree of the rational curves. Let $\varrho : \mathcal{H} \to \operatorname{Spec}(R)$ be the structural morphism: the fiber of ϱ over a point which corresponds to a maximal ideal is a scheme $H_{R/\mathfrak{m}}$ defined over the field (of positive characteristic) $R/\mathfrak{m}$. Since by step 1 the theorem holds for $R/\mathfrak{m}$ we have that the scheme $H_{R/\mathfrak{m}}$ is nonempty.

$$\begin{tikzcd} H_{R/\mathfrak{m}} \arrow[r] \arrow[d] & \mathcal{H} \arrow[d, "\varrho"] \\ \operatorname{Spec}(R/\mathfrak{m}) \arrow[r] & \operatorname{Spec}(R) \end{tikzcd}$$

In particular, the image of ϱ contains all the closed points of $\operatorname{Spec}(R)$.

Definition 2.2.3. A Zariski topological space Z is a noetherian topological space such that every closed irreducible subset $Y \subseteq Z$ has a unique generic point, i.e. a unique point ξ satisfying $\overline{\{\xi\}} = Y$.

Definition 2.2.4. Let X be a topological space. A subset of X is called constructible if it belongs to a family $\mathcal{F}$ which has the following properties:

(a) $\mathcal{F}$ contains the open subsets of X;
(b) if $Y_1, \dots, Y_k \in \mathcal{F}$ then $\bigcap_{j=1,\dots,k} Y_j \in \mathcal{F}$;
(c) if $Y \in \mathcal{F}$ then $Z \setminus Y \in \mathcal{F}$.

Proposition 2.2.5. *(a) A subset of a topological space X is constructible if and only if it can be written as the finite union of disjoint locally closed subsets of X;*
(b) a constructible subset of a Zariski topological space Z is dense if and only if it contains the generic point of Z.

The following theorem is due to Chevalley:

Theorem 2.2.6. *Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. Then the image of every constructible subset of X is constructible in Y.*

In our case, $\mathcal{H}$ is a quasi-projective scheme, hence it is constructible; by Chevalley's theorem its image in $\operatorname{Spec}(R)$ via ϱ is also constructible, and by lemma 2.2.2 it is dense in $\operatorname{Spec}(R)$. Then proposition 2.2.5 yields that it contains the generic point of $\operatorname{Spec}(R)$: this means that the generic fiber of ϱ (i.e. the fiber corresponding to $K(R)$) is nonempty, and the theorem is proved. □

We now give a generalization of the bend & break lemma: given a polarization H on X and a curve which deforms nontrivially keeping some points fixed, we produce a rational curve passing through the fixed points and give an upper bound on its H-degree.

Lemma 2.2.7. *[Deb01, Theorem 3.5] Let X be a projective variety and let H be an ample divisor on X. Let $f : C \to X$ be a smooth curve and let B be a finite nonempty subset of C. If*

$$\dim_{[f]} \operatorname{Hom}(C, X; f_{|B}) \geq 1$$

then there exists on X a rational curve Γ that meets $f(B)$ and such that

$$H \cdot \Gamma \leq 2 \frac{H \cdot C}{\operatorname{Card}(B)}.$$

Using this lemma we can improve the result of theorem 2.2.1:

Theorem 2.2.8. *[Deb01, Theorem 3.6] Let X be a smooth projective variety, let H be an ample divisor on X and let $f : C \to X$ be a smooth curve such that $-K_X \cdot C > 0$. Given any point x on $f(C)$ there exists a rational curve Γ through x with*

$$H \cdot \Gamma \leq 2 \dim X \frac{H \cdot C}{-K_X \cdot C}.$$

2.3 The Cone theorem

In this section we prove a fundamental result about the structure of the Mori cone of a smooth projective variety X; more precisely, it states that $\mathrm{NE}(X)$ is locally polyhedral in the part contained in $\mathrm{N}_1(X)_{K_X<0}$. This result is known as the Cone theorem, and it was proved by Mori as an application of his bend & break results.

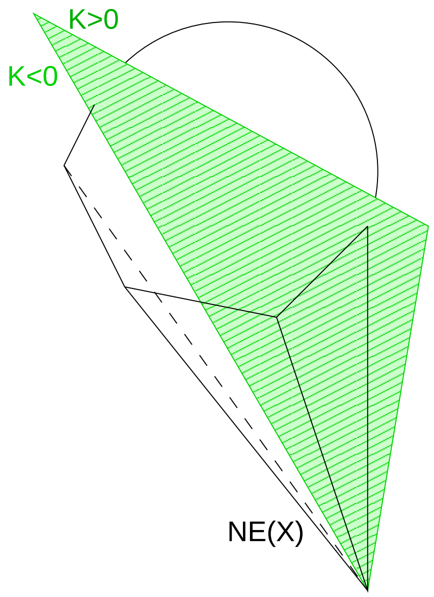
Theorem 2.3.1 (Cone theorem). *Let X be a smooth projective variety of dimension n. Then there exist on X countably many rational curves $\{C_i\}_{i\in\mathbb{N}}$ such that*

$$0 < -K_X \cdot C_i \leq n+1$$

and

$$\overline{\mathrm{NE}(X)} = \overline{\mathrm{NE}(X)}_{K_X \geq 0} + \sum_{i\in\mathbb{N}} \mathbb{R}^+[C_i].$$

Moreover, the rays $R_i = \mathbb{R}^+[C_i]$ are locally discrete in the half-space of $\mathrm{N}_1(X)$ given by $\{z \in \mathrm{N}_1(X) \mid K_X \cdot z < 0\}$.



Proof. By remark 1.4.2 we know that on X there exist only countably many numerical classes of rational curves; for each class of anticanonical degree $\leq n+1$ we pick a representative C_i.

Step 1. *The rays $\mathbb{R}^+[C_i]$ are locally discrete in $\mathrm{N}_1(X)_{K_X<0}$.*
Let H be an ample divisor on X. Since $\mathrm{N}_1(X)_{K_X<0}$ can be written as

$$\mathrm{N}_1(X)_{K_X<0} = \bigcup_{\epsilon>0} \mathrm{N}_1(X)_{K_X+\epsilon H<0},$$

it is enough to show that for a fixed $\epsilon > 0$ the half-space $\mathrm{N}_1(X)_{K_X+\epsilon H<0}$ contains only finitely many classes among the $[C_i]$.
If $(K_X + \epsilon H) \cdot C_i < 0$ then

$$H \cdot C_i < \frac{1}{\epsilon}(-K_X \cdot C_i) \leq \frac{1}{\epsilon}(n+1),$$

and the second part of Kleiman's criterion (1.1.11) proves the claim.

Step 2. *$\overline{\mathrm{NE}(X)}$ is the closure of $V = \overline{\mathrm{NE}(X)}_{K_X \geq 0} + \sum \mathbb{R}^+[C_i]$*

Suppose that this is not the case. Since by corollary 1.1.13 $\mathrm{NE}(X)$ contains no lines, it is an elementary property of cones (see [Deb01, Lemma 6.7 (d)]) that there exists an $\mathbb{R}$-divisor $M \in \mathrm{N}^1(X)$ such that M

is nonnegative on $\overline{\mathrm{NE}(X)}$,
is positive on $\overline{V} \setminus \{0\}$,
vanishes at some nonzero point z of $\overline{\mathrm{NE}(X)}$.

Clearly the point z cannot belong to $\overline{\mathrm{NE}(X)}_{K_X \geq 0}$, so $K_X \cdot z < 0$.

Choose a norm on $\mathrm{N}_1(X)$ such that each irreducible curve $C \subset X$ satisfies $\|C\| \geq 1$ (this is possible since the numerical classes of irreducible curves are discrete in $\mathrm{N}_1(X)$) and assume, up to replace M with a multiple, that $M \cdot v \geq 2\|v\|$ for every $v \in \overline{V}$. Since the class of M in $\mathrm{N}^1(X)$ is a limit of classes of ample $\mathbb{Q}$-divisors, and z is a limit in $\mathrm{N}_1(X)$ of classes of effective rational 1-cycles, we can find an ample $\mathbb{Q}$-divisor H and an effective rational 1-cycle Z such that

$$2n(H \cdot Z) < -K_X \cdot Z \qquad \text{and} \qquad H \cdot v \geq \|v\| \tag{2.1}$$

for every $v \in \overline{V}$. Moreover, up to throw away the other components, we can assume that every component C of Z satisfies $-K_X \cdot C > 0$.

Since the class of every rational curve $\Gamma \subset X$ of anticanonical degree $\leq n+1$ is in $\overline{V}$, we have that in our assumptions $H \cdot \Gamma \geq 1$, and theorem 2.2.8 implies that

$$2n \frac{H \cdot C}{-K_X \cdot C} \geq H \cdot \Gamma \geq 1$$

for every component C of Z, contradicting the first inequality in (2.1).

Step 3. *For any set of indices J, the cone $V_J = \overline{\mathrm{NE}(X)}_{K_X \geq 0} + \sum_{j \in J} \mathbb{R}^+[C_j]$ is closed.*

Since $\overline{\mathrm{NE}(X)}$ does not contain lines, it coincides with the convex hull of its extremal rays (see [Deb01, Lemma 6.7 (b)]); then it is enough to show that every extremal ray $\mathbb{R}^+ z$ in $\overline{V}_J$ satisfying $K_X \cdot z < 0$ is in V_J.

Write z as the limit of a sequence $\{z_m + w_m\}$, where $K_X \cdot z_m \geq 0$ and $w_m \in \sum_J \mathbb{R}^+[C_j]$, and fix an ample divisor H on X; since the numerical sequences $\{H \cdot z_m\}$

and $\{H \cdot w_m\}$ are bounded (for example by $H \cdot z + 1$), up to extract subsequences they have limits in $\overline{V}_J$. But z spans an extremal ray, so these limits have to be nonnegative multiples of z; moreover we know that $K_X \cdot z < 0$, so the limit of $\{z_m\}$ must be zero.

Fix a positive ϵ such that $(K_X + \epsilon H) \cdot z < 0$; by step 1, there are only finitely many classes $[C_{j_1}], \ldots, [C_{j_k}]$ with $j_\alpha \in J$ such that $(K_X + \epsilon H) \cdot C_{j_\alpha} < 0$. If we write w_m as

$$w_m = w'_m + w''_m$$

with $w'_m \in \langle [C_{j_1}], \ldots, [C_{j_k}] \rangle$ and $(K_X + \epsilon H) \cdot w''_m \geq 0$, we can assume as before that both sequences $\{w'_m\}$ and $\{w''_m\}$ converge to some nonnegative multiples of z, and in particular that $\{w''_m\}$ converges to zero. This proves that z is the limit of a linear combination with rational coefficients of $[C_{j_1}], \ldots, [C_{j_k}]$, and hence that z belongs to V_J.

Conclusion. Let I be the set of indices i such that $\mathbb{R}^+[C_i]$ is an extremal ray of $\mathrm{NE}(X)$. The proof of step 3 actually shows that every extremal ray of $\mathrm{NE}(X)$ is proportional to some $[C_i]$, and this concludes the proof of the Cone theorem. □

Definition 2.3.2. A subcone σ of $\mathrm{NE}(X)$ is called an extremal face if it satisfies the following condition:

$$\text{if} \quad a, b \in \mathrm{NE}(X) \quad \text{and} \quad a + b \in \sigma \quad \text{then} \quad a, b \in \sigma.$$

An extremal face of dimension one is called an extremal ray, and a curve whose numerical class belongs to an extremal ray is called an extremal curve.

Definition 2.3.3. An extremal face σ of $\mathrm{NE}(X)_{K_X<0}$ is called a negative extremal face of $\mathrm{NE}(X)$; a negative extremal face of dimension one is called a negative extremal ray.

2.4 Fano-Mori contractions

Definition 2.4.1. A contraction $f : X \to Y$ is a proper morphism with connected fibers between two normal varieties X and Y.

If X and Y are projective varieties, we can consider the convex subcone $\mathrm{NE}(f)$ of $\mathrm{NE}(X)$ generated by the classes of curves contracted by f. Since Y is projective, the projection formula yields that being contracted by f is a numerical property, and $\mathrm{NE}(f)$ is nothing but the intersection of $\mathrm{NE}(X)$ with the vector space $\ker(f_*)$. Moreover $\mathrm{NE}(f)$ is extremal and it determines uniquely the contraction f up to isomorphism (see [Deb01, Proposition 1.14]).

Part of Mori's program aims at giving conditions under which a given subcone V of $\mathrm{NE}(X)$ can be contracted. This is the content of the Contraction theorem.

Definition 2.4.2. A Fano-Mori contraction $\varphi : X \to Y$ of a smooth variety X is a contraction such that the anticanonical divisor $-K_X$ is φ-ample, that is to say the divisor $(-K_X)_{|F}$ is ample on a general fiber F of φ.

Remark 2.4.3. In this case the cone $\mathrm{NE}(\varphi)$ is a negative extremal face of $\mathrm{NE}(X)$.

We will not give the whole proof of the Contraction theorem (see for example [Deb01, Theorem 7.39]), but we state here the two results on which it is based and which are due to Kawamata: the Rationality theorem and the Base Point Free theorem.

Theorem 2.4.4 (Rationality Theorem). *[Deb01, Theorem 7.34] Let X be a smooth complex projective variety such that K_X is not nef. Let H be a nef and big Cartier divisor on X. Then the number*

$$r = \sup\{t \in \mathbb{R} \mid H + tK_X \text{ is nef}\}$$

is rational.

Corollary 2.4.5. *Let X be a smooth projective variety and let σ be a negative extremal face of* $\mathrm{NE}(X)$. *Then there exists a nef divisor H on X such that*

$$\sigma = \{Z \in \mathrm{NE}(X) \mid H \cdot Z = 0\},$$

and the divisor $mH - K_X$ is ample for all integers $m \gg 0$.
The divisor H is called a supporting divisor *of the face σ.*

Theorem 2.4.6 (Base-point free theorem). *[Deb01, Theorem 7.32] Let X be a smooth variety and let H be a nef divisor on X such that $aH - K_X$ is nef and big for some positive integer a. Then the linear system $|mH|$ is base-point free for all integers $m \gg 0$.*

Combining these two results, we have that to a negative extremal face σ of $\mathrm{NE}(X)$ we can associate a nef divisor H, one multiple of which induces a morphism $\varphi_{|mH|} : X \to Y \subseteq \mathbb{P}^N$. The part with connected fibers of the Stein factorization of $\varphi_{|mH|}$ is a Fano-Mori contraction; namely the following theorem holds:

Theorem 2.4.7 (Contraction theorem). *[KM98] Let X be a smooth variety and let H be a nef divisor on X such that*

$$\sigma := H^{\perp} \cap \mathrm{NE}(X)$$

is entirely contained in $\mathrm{N}_1(X)_{K_X<0}$. Then there exists a projective morphism

$$\varphi : X \longrightarrow Y$$

onto a normal projective variety Y, which is characterized by the following properties:

(a) a curve $C \subset X$ is contracted to a point by φ if and only if $H \cdot C = 0$;
(b) φ has connected fibers;
(c) $H = \varphi^* A$ for some ample Cartier divisor $A \in \mathrm{Div}(Y)$.

Definition 2.4.8. The map φ of the above theorem is usually called the Fano-Mori contraction (or the extremal contraction) associated to the face σ. A Cartier divisor H such that $H = \varphi^* A$ for some ample divisor A on Y is called a good supporting divisor of the map φ (or of the face σ).

Notation. Since the Cone and Contraction theorem give us no informations about the positive part $\mathrm{NE}(X)_{K_X \geq 0}$ of $\mathrm{NE}(X)$, we will focus our attention on negative extremal faces and rays, and from now on we will simply call them “extremal”.

Definition 2.4.9. We denote with

$$E = E(\varphi) := \{x \in X \mid \dim(\varphi^{-1}\varphi(x)) > 0\}$$

the exceptional locus of φ; it coincides with the union of all curves in X which are contracted by φ, and for this reason it is sometimes called the locus of σ.

Definition 2.4.10. An extremal ray R of $\mathrm{NE}(X)$ is called

numerically effective, or of fiber type, if $\dim Y < \dim X$; in this case of course $E = X$;

non nef, or birational, if $\dim Y = \dim X$; the terminology is due to the fact that if R is non nef then there exists an irreducible divisor D_R which is negative on curves in R (see the proof of [KMM87, Proposition 5.1.6]).
Moreover, if the codimension of E is equal to one the ray and the associated contraction are called divisorial, otherwise they are called small.

Fano-Mori contractions of smooth threefolds were classified by Mori in [Mor82], while the case of smooth fourfolds was investigated by Andreatta and Wiśniewski in [AW98].

3

Families of rational curves

3.1 Families of rational curves

Definition 3.1.1. We define a family of rational curves to be an irreducible component $V \subset \mathrm{Ratcurves}^n(X)$.
Given a rational curve $f : \mathbb{P}^1 \to C \subset X$ we will call a family of deformations of f (or of C) any irreducible component $V \subset \mathrm{Ratcurves}^n(X)$ containing $u(f)$.

Given a family V of rational curves, we can consider the subscheme given by the intersection $V \cap \mathrm{Ratcurves}^n(X, x)$, which parametrizes curves in V passing through x. We usually denote by V_x a component of this subscheme.

Diagram 1.1 can be restricted to a family V, and we obtain the basic diagram

$$\begin{tikzcd} p^{-1}(V) =: U \arrow[r, "i"] \arrow[d, "p"'] & X \\ V & \end{tikzcd} \tag{3.1}$$

where i is the map induced by the evaluation $e : \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to X$ and p is a $\mathbb{P}^1$-bundle.

Definition 3.1.2. We define the locus of the family V to be the closure of the image of U in X, and we denote it by $\mathrm{Locus}(V)$; we say that V dominates a closed subset $Y \subseteq X$ if $\mathrm{Locus}(V) = Y$, and in the particular case when $Y = X$ we say that V is a covering family.

Definition 3.1.3. Given a divisor D on X, we will denote by $\deg_D V$ the integer $\deg_D C := D \cdot C$ for any curve $C \in V$.
For simplicity of notation, we will set $\deg V := \deg_{-K_X} V$, and we will call it the anticanonical degree (or the degree, if no confusion can arise) of the family V.

Remark 3.1.4. Let H be an ample divisor on X, and let $Y \subseteq X$ be a closed irreducible subset such that for every point $y \in Y$ there exists a rational curve C_y with $\deg_H C_y \leq d$. Then there exists a family V which dominates Y and such that $\deg_H V \leq d$.

Proof. By remark 1.4.2 it follows that on X there exist only finitely many families of rational curves of degree $\leq d$; since Y is contained in the union of their loci and the base field is uncountable, then Y must be contained in $\mathrm{Locus}(V)$ for some family V. □

3.2 Minimizing families of rational curves

Definition 3.2.1. Let $V \subseteq \mathrm{Ratcurves}^n(X)$ be a family of rational curves on X. Then

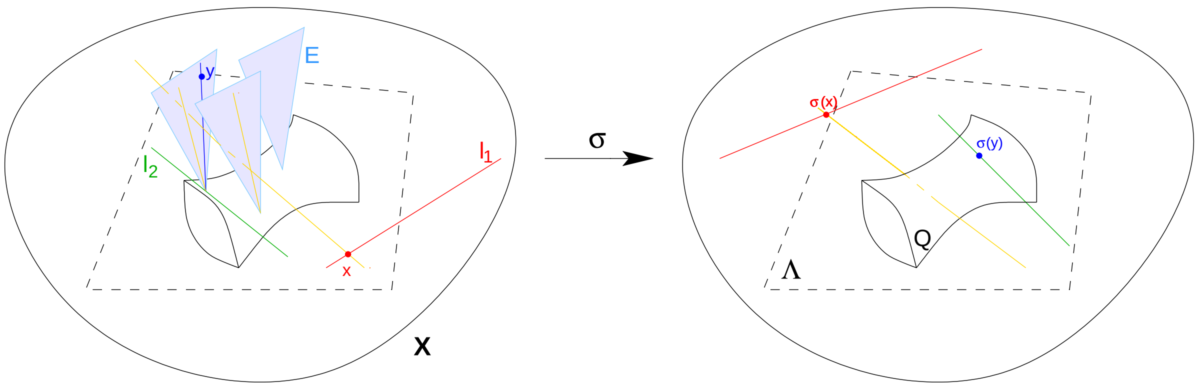
(a) V is unsplit if it is proper;
(b) V is locally unsplit if for the general $x \in \mathrm{Locus}(V)$ every component V_x of $V \cap \mathrm{Ratcurves}^n(X, x)$ is proper;
(c) V is generically unsplit if there is at most a finite number of curves of V passing through two general points of $\mathrm{Locus}(V)$.

Remark 3.2.2. Definition (a) has a simple geometric interpretation: in fact, the scheme $\mathrm{Ratcurves}^n(X)$ has a natural inclusion into the scheme $\mathrm{Chow}_1(X)$ which we introduced in theorem 1.1.4 (this is an easy consequence of the definition of $\mathrm{Ratcurves}^n(X)$ as given for instance in [Kol96, II.2.11]).
Let W be the image of V in $\mathrm{Chow}_1(X)$: then V is proper in $\mathrm{Ratcurves}^n(X)$ if and only if W is closed in $\mathrm{Chow}_1(X)$. A point in $\overline{W} \setminus W$ corresponds to a 1-cycle $\sum a_i[C_i]$, where C_i are (irreducible) rational curves on X, $a_i \in \mathbb{N}$ and $\sum a_i \geq 2$. Thus if the family V is not unsplit the general rational curve in V degenerates into a reducible 1-cycle.

Remark 3.2.3. Note that $(a) \Rightarrow (b) \Rightarrow (c)$. In particular, the second implication follows from theorem 2.1.2.

Remark 3.2.4. If R is an extremal ray of $\mathrm{NE}(X)$ and C is an extremal curve such that $[C] \in R$ and $\deg C$ is minimal among curves whose class belongs to R (we will call C a minimal rational extremal curve), then any family of deformations of C is unsplit. In fact, if this were not the case, then by remark 3.2.2 C would degenerate into a reducible cycle $\sum a_i[C_i]$, and since R is an extremal ray we would have that $[C_i] \in R$ for every i, against the minimality of $\deg C$.

Example 3.2.5. Let $\sigma : X \to \mathbb{P}^5$ the blow-up of $\mathbb{P}^5$ along a two-dimensional quadric Q, let Λ be the three-dimensional linear subspace of $\mathbb{P}^5$ which contains Q and let E be the exceptional divisor of σ.
Let $l_1 \subset X$ be the strict transform of a line in $\mathbb{P}^5$ which does not meet Q, and let V^1 be a family of deformations of l_1: then V^1 is generically unsplit but it is not locally unsplit. In fact, if we consider two points x, $y \in l_1$, the only curve in V^1 which passes through x and y is l_1; on the other hand, for every $x \in \text{Locus}(V^1)$ curves in V^1_x degenerate into a reducible cycle, whose components are the strict transform of a line $l \subset \mathbb{P}^5$ through $\sigma(x)$ which intersects Q and a line in each fiber of σ over the points $l \cap Q$.
Now let $l_2 \subset X$ be the strict transform of a line in $\mathbb{P}^5$ which intersects Q in one point, and let V^2 be a family of deformations of l_2: then V^2 is locally unsplit but it is not unsplit. In fact, for general $x \in X$ curves in V^2_x do not degenerate into reducible cycles, but if we consider a point $y \in E$ or in the strict transform of Λ, we have that curves in V^2_y degenerate into the strict transform of a line in Λ through $\sigma(y)$ and a line in the fiber of σ which contains y.



Definition 3.2.6. Let H be an ample divisor on X and let $Y \subseteq X$ be a closed subset. A family of rational curves V is a minimal dominating family for Y if V dominates Y and $\deg_H V$ is minimal among all families which dominate Y.

Remark 3.2.7. A minimal dominating family for a closed subset Y is locally unsplit: in fact, if for the general $x \in \text{Locus}(V)$ the pointed family V_x were non unsplit, then there would exist a rational curve C through x such that $\deg_H C < \deg_H V$. But in this case, by remark 3.1.4, there would exist a family of rational curves which dominates Y and whose degree with respect to H is strictly less than $\deg_H V$, against the minimality of V.

Proposition 3.2.8. *Let X be a projective variety and V a family of rational curves on X. Assume either that V is generically unsplit and x is a general point in* $\mathrm{Locus}(V)$ *or that V is unsplit and x is any point in* $\mathrm{Locus}(V)$. *Then*

$$\dim V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) - 2.$$

Proof. Take a point $x \in \mathrm{Locus}(V)$ and consider the diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "p"'] & \mathrm{Locus}(V) \subseteq X \\ V & \end{tikzcd}$$

then $V_x = \{[f] \in V \mid f(0) = x\} = p(i^{-1}(x))$, and since i does not contract any fiber of p we have

$$\dim V_x = \dim i^{-1}(x);$$

upper semi-continuity of the fiber dimension implies that

$$\dim i^{-1}(x) \geq \dim U - \dim \mathrm{Locus}(V)$$

(and equality holds for general x), so

$$\dim V_x \geq \dim U - \dim \mathrm{Locus}(V) = \dim V - \dim \mathrm{Locus}(V) + 1.$$

Now take $y \in \mathrm{Locus}(V_x)$ and consider the pointed version of the previous diagram

$$\begin{tikzcd} U_x \arrow[r, "i_x"] \arrow[d, "p_x"'] & \mathrm{Locus}(V_x) \subseteq X \\ V_x & \end{tikzcd}$$

arguing as before we have

$$\dim V_{x,y} = \dim i_x^{-1}(y) \geq \dim V_x - \dim \mathrm{Locus}(V_x) + 1.$$

The two inequalities together give

$$\begin{aligned}\dim V_{x,y} &\geq \dim V_x - \dim \mathrm{Locus}(V_x) + 1 \\ &\geq \dim V - \dim \mathrm{Locus}(V) - \dim \mathrm{Locus}(V_x) + 2.\end{aligned}$$

Now, if V is unsplit or if V is generically unsplit and x, y are general points in $\mathrm{Locus}(V)$ we have that $\dim V_{x,y} = 0$, so we conclude that

$$\dim V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) - 2;$$

note also that equality holds if V is generically unsplit and x is a general point in $\mathrm{Locus}(V)$. □

Proposition 3.2.9 (Ionescu-Wiśniewski inequality). *Let X be a smooth projective variety and V a family of rational curves on X. Assume either that V is generically unsplit and x is a general point in* $\mathrm{Locus}(V)$ *or that V is unsplit and x is any point in* $\mathrm{Locus}(V)$. *Then*

(a) $\dim X + \deg V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1$;
(b) $\deg V \leq \dim \mathrm{Locus}(V_x) + 1$.

Proof. The proof of (a) is an easy consequence of proposition 3.2.8 and of theorem 1.3.1 (b), noting that if V is the family of deformations of f then

$$\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, X) = \dim V + \dim \mathrm{Aut}(\mathbb{P}^1) = \dim V + 3.$$

(b) follows from the obvious observation that $\dim \mathrm{Locus}(V) \leq \dim X$. □

Remark 3.2.10. It follows immediately from the above proposition that if V is locally unsplit and $\deg V = \dim \mathrm{Locus}(V_x) + 1$ for general $x \in \mathrm{Locus}(V)$ then V is a covering family.

Proposition 3.2.11 (Fiber locus inequality). *Let φ be a Fano-Mori contraction of X and let $E = E(\varphi)$ be its exceptional locus; let F be an irreducible component of a (non trivial) fiber of φ. Then*

$$\dim E + \dim F \geq \dim X + l - 1$$

where

$$l = \min\{-K_X \cdot C \mid C \textit{ is a rational curve in } F\}.$$

If φ is the contraction of an extremal ray R, then $l =: l(R)$ is called the length of the ray.

Proof. The proof follows directly from the Ionescu-Wiśniewski inequality, noting that if V is the family of deformations of a rational curve C which is contained in F then E contains $\mathrm{Locus}(V)$ and F contains $\mathrm{Locus}(V_x)$ for some point $x \in X$. □

3.3 Extremal rays of high length

In this section we give a short overview on the characterization of smooth projective varieties which admit an extremal ray R of high length:

$l(R) = n + 1$
It has been proved recently by Cho, Miyaoka and Shepherd-Barron [CMSB02] and by Kebekus [Keb02] that the existence of an extremal ray of length $n+1$ is in fact a characterization of the projective space $\mathbb{P}^n$. More precisely, Kebekus proved that $\mathbb{P}^n$ is the only projective variety which admits a covering locally unsplit family of rational curves which has anticanonical degree $= n + 1$.

$l(R) = n$
In this case, either X has Picard number $\rho_X = 1$, and it is conjectured that in this case $X = \mathbb{Q}^n$, or $\rho_X = 2$ and X is a $\mathbb{P}^{n-1}$-bundle onto a smooth curve (see [Wiś89]).

$l(R) = n - 1$
It has been proved by Andreatta and Occhetta [AO02] that there exists in $\mathrm{NE}(X)$ a non nef extremal ray R of length $n-1$ if and only if X is the blow-up of a smooth projective variety Y at a point.
This result is a consequence of the following more general theorem:

Theorem 3.3.1. *[AO02, Theorem 5.1] Let X be a smooth projective variety of dimension n over an algebraically closed field of characteristic zero. The two following facts are equivalent:*

(a) there exists an extremal ray R such that the associated contraction is divisorial and the fibers have dimension $l(R)$;
(b) X is the blow-up of a smooth projective variety Y along a smooth subvariety of codimension $l(R) + 1$.

Finally, if there exists in $\mathrm{NE}(X)$ a non nef extremal ray of length $n - 2$, one of the following cases occurs:

(a) $\varphi(E)$ is a point and $(E, -E_{|E}) \simeq (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}}(2))$;
(b) $\varphi(E)$ is a point and $(E, -E_{|E}) \simeq (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}}(1))$, where $\mathbb{Q}^{n-1}$ is a possibly singular quadric;
(c) φ is the blow-up of a smooth variety Y along a smooth curve $\varphi(E) \subset Y$,

where φ is the contraction associated to R and E its exceptional locus [AO02].

3.4 Chow families

Definition 3.4.1. We define a Chow family of rational curves to be an irreducible component $\mathcal{V} \subset \mathrm{Chow}_1(X)$ (see definition 1.1.4) parametrizing rational connected 1-cycles.

Given a Chow family of rational curves, we have a diagram as before, coming from the universal family over $\mathrm{Chow}_1(X)$.

$$\begin{tikzcd} \mathcal{U} \arrow[r, "i"] \arrow[d, "p"'] & X \\ \mathcal{V} & \end{tikzcd} \tag{3.2}$$

In the diagram i is the map induced by the evaluation and the fibers of p are connected and have rational components. Both i and p are proper (see for instance [Kol96, II.2.2]). By [Kol96, IV.4.10] the family $\mathcal{V}$ defines a proper prerelation in the sense of [Kol96, IV.4.6] (note that schemes and morphisms appearing in that definition are those of the normal form [Kol96, IV.4.4.5]).

Note that we can define the locus and the degree of a Chow family in the same way as we have done for families of rational curves. Note also that if $\mathcal{V}$ is a Chow family then $\mathrm{Locus}(\mathcal{V})$ is the image of $\mathcal{U}$ in X in diagram 3.2, so, since $\mathcal{V}$, p and i are proper, $\mathrm{Locus}(\mathcal{V})$ is a closed subset of X.

Definition 3.4.2. If V is a family of rational curves we can consider the closure of the image of V in $\mathrm{Chow}_1(X)$, and call it the Chow family associated to V.

Remark 3.4.3. If V is proper, i.e. if the family is unsplit, then V corresponds to the normalization of the associated Chow family $\mathcal{V}$; in particular V itself defines a proper prerelation. For this reason we will not use calligraphic letters when dealing with unsplit families.

Definition 3.4.4. Let $\mathcal{V}$ be the Chow family associated to a family of rational curves V. We say that V is quasi-unsplit if every component of any reducible cycle in $\mathcal{V}$ is numerically proportional to V.

Example 3.4.5. Let $R = \mathbb{R}^+[C]$ be an extremal ray of X, let φ_R be the associated contraction and let E be the exceptional locus of φ_R.
Then through every point of E there exists a rational curve whose numerical equivalence class is contained in R. In fact, through every point of E there exists a curve C which is contracted by φ_R; if such a curve is nonrational theorem 2.1.1 implies that $C \equiv C' + Z$ for some rational 1-cycle $Z = \sum a_i \Gamma_i$, and since C is extremal we have that $[\Gamma_i] \in R$ for every i.
We can thus find a family V_R which dominates E and has minimal degree; such a family is locally unsplit and quasi-unsplit, since if a curve in V_R degenerates into a reducible cycle then all of its components must belong to R.

3.5 Chains of rational curves

Let X be a normal proper variety, $\mathcal{V}^1, \dots, \mathcal{V}^k$ Chow families of rational curves on X and Y a subset of X.

Definition 3.5.1. We denote by $\mathrm{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ the set of points $x \in X$ such that there exist cycles $C_1, \dots, C_k$ with the following properties:

C_i belongs to the family $\mathcal{V}^i$;
$C_i \cap C_{i+1} \neq \emptyset$;
$x \in C_1 \cup \dots \cup C_k$,

i.e. $\mathrm{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ is the set of points which belong to a connected chain of k cycles belonging *respectively* to the families $\mathcal{V}^1, \dots, \mathcal{V}^k$.

Definition 3.5.2. We denote by $\mathrm{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ the set of points $x \in X$ such that there exist cycles $C_1, \dots, C_k$ with the following properties:

C_i belongs to the family $\mathcal{V}^i$;
$C_i \cap C_{i+1} \neq \emptyset$;
$C_1 \cap Y \neq \emptyset$ and $x \in C_k$,

i.e. $\mathrm{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ is the set of points that can be joined to Y by a connected chain of k cycles belonging *respectively* to the families $\mathcal{V}^1, \dots, \mathcal{V}^k$.

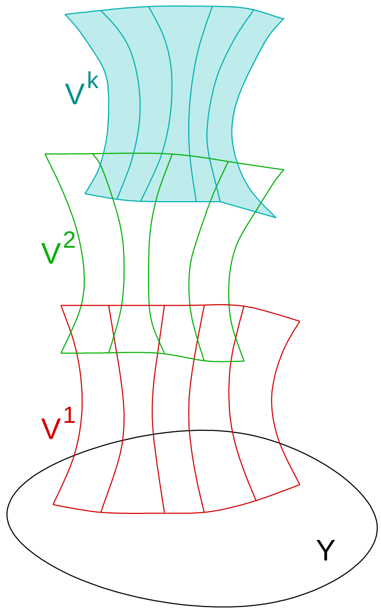


Fig. 3.1. $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$

Note that $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y \subset \mathrm{Locus}(\mathcal{V}^k)$.

Remark 3.5.3. If Y is a closed subset, then $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ is closed. We prove the statement by induction, since we have

$$\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y = \mathrm{Locus}(\mathcal{V}^k)_{\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^{k-1})_Y}.$$

With the notation of diagram 3.2 let $\mathcal{V}_Y = p(i^{-1}(Y \cap \mathrm{Locus}(\mathcal{V})))$ be the subset of $\mathcal{V}$ parametrizing cycles of $\mathcal{V}$ meeting Y; $\mathrm{Locus}(\mathcal{V})_Y$ is just $i(p^{-1}(\mathcal{V}_Y))$, so it is closed by the properness of i and p.

If we consider unsplit families $V^1, \ldots, V^k$, we are able to provide some estimates for the dimension of $\mathrm{Locus}(V^1, \ldots, V^k)_x$:

Theorem 3.5.4. *[BCDD03, Théorème 5.2] Let $V^1, \ldots, V^k$ be unsplit families of rational curves on X. If the corresponding classes in $\mathrm{N}_1(X)$ are independent, then either* $\mathrm{Locus}(V^1, \ldots, V^k)_x$ *is empty or it has dimension greater or equal to* $\sum \deg V^i - k$.

Using the same techniques as in the proof of theorem 3.5.4 we obtained the following:

Lemma 3.5.5. *Let $Y \subset X$ be a closed subset and V an unsplit family. Assume that curves contained in Y are numerically independent from curves in V, and that $Y \cap \operatorname{Locus}(V) \neq \emptyset$. Then for a general $y \in Y \cap \operatorname{Locus}(V)$*

(a) $\dim \operatorname{Locus}(V)_Y \geq \dim(Y \cap \operatorname{Locus}(V)) + \dim \operatorname{Locus}(V_y)$;
(b) $\dim \operatorname{Locus}(V)_Y \geq \dim Y + \deg V - 1$.

Moreover, if $V^1, \dots, V^k$ are numerically independent unsplit families such that curves contained in Y are numerically independent from curves in $V^1, \dots, V^k$ then either $\operatorname{Locus}(V^1, \dots, V^k)_Y = \emptyset$ *or*

(c) $\dim \operatorname{Locus}(V^1, \dots, V^k)_Y \geq \dim Y + \sum \deg V^i - k$.

Proof. We refer to diagram 3.1. Since V is unsplit, for a point y in $Y \cap \operatorname{Locus}(V)$ we have

$$\dim i^{-1}(y) = \dim V_y = \dim \operatorname{Locus}(V_y) - 1.$$

So, setting $V_Y = p(i^{-1}(Y))$ and $U_Y = p^{-1}(V_Y)$, we have for general $y \in Y \cap \operatorname{Locus}(V)$,

$$\begin{aligned}\dim U_Y &= \dim(Y \cap \operatorname{Locus}(V)) + \dim \operatorname{Locus}(V_y) \geq \\ &\geq \dim Y + \dim \operatorname{Locus}(V) - n + \dim \operatorname{Locus}(V_y) \geq \\ &\geq \dim Y + \deg V - 1.\end{aligned}$$

Since $\operatorname{Locus}(V)_Y = i(U_Y)$, (a) and (b) will follow if we prove that $i : U_Y \to X$ is generically finite.
To show this we take a point $x \in i(U_Y) \backslash Y$ and we suppose that $i^{-1}(x) \cap U_Y$ contains a curve C' which is not contained in any fiber of p; let B' be the curve $p(C') \subset V_Y$ and let $\nu : B \to B'$ be the normalization of B'.
By base change we obtain the following diagram

$$\begin{tikzcd} S_B \arrow[r, "j"] \arrow[d, "p_B"'] & X \\ B & \end{tikzcd}$$

Let C_Y be a curve in S_B which dominates B and whose image via j is contained in Y; such a curve exists since the image via j of every fiber of p_B meets Y. Now two cases are possible: either $j(C_Y)$ is a point, and therefore we have a one-parameter family of curves passing through two fixed points, contradicting the fact that V is unsplit (see for instance [Kol96, IV.2.3]) or $j(C_Y)$ is a curve in $Y \cap \operatorname{Locus}(V_y)$, so a

curve in Y is numerically proportional to a curve parametrized by V, against the assumptions.
To show (c) it is enough to recall that, as already observed in remark 3.5.3, we have $\mathrm{Locus}(\mathcal{V}^1,\ldots,\mathcal{V}^k)_Y = \mathrm{Locus}(\mathcal{V}^k)_{\mathrm{Locus}(\mathcal{V}^1,\ldots,\mathcal{V}^{k-1})_Y}$. □

Remark 3.5.6. If in the previous theorem V^1 is not a covering family and $\mathrm{Locus}(V^1,\ldots,V^k)_x$ is nonempty, then

$$\dim \mathrm{Locus}(V^1,\ldots,V^k)_x \geq \sum \deg V^i - k + 1.$$

In fact $\mathrm{Locus}(V^1,\ldots,V^k)_x = \mathrm{Locus}(V^2,\ldots,V^k)_{\mathrm{Locus}(V^1_x)}$, and we can apply part (c) of lemma 3.5.5, recalling that $\dim \mathrm{Locus}(V^1_x) = \deg V^1 - 1$ implies that V^1 is covering (remark 3.2.10).

Definition 3.5.7. We denote by $\mathrm{ChLocus}_m(\mathcal{V}^1,\ldots,\mathcal{V}^k)_Y$ the set of points $x \in X$ such that there exist cycles $C_1,\ldots,C_m$ with the following properties:

C_i belongs to a family $\mathcal{V}^j$;
$C_i \cap C_{i+1} \neq \emptyset$;
$C_1 \cap Y \neq \emptyset$ and $x \in C_m$,

i.e. $\mathrm{ChLocus}_m(\mathcal{V}^1,\ldots,\mathcal{V}^k)_Y$ is the set of points that can be joined to Y by a connected chain of at most m cycles belonging to the families $\mathcal{V}^1,\ldots,\mathcal{V}^k$.

Remark 3.5.8. Note that

$$\mathrm{ChLocus}_m(\mathcal{V}^1,\ldots,\mathcal{V}^k)_Y = \bigcup_{1\leq i(j)\leq k} \mathrm{Locus}(\mathcal{V}^{i(1)},\ldots,\mathcal{V}^{i(m)})_Y;$$

in particular, if Y is a closed subset then $\mathrm{ChLocus}_m(\mathcal{V}^1,\ldots,\mathcal{V}^k)_Y$ is closed.

Definition 3.5.9. We define on X a relation of rational connectedness with respect to $\mathcal{V}^1,\ldots,\mathcal{V}^k$ in the following way: x and y are in the $\mathrm{rc}(\mathcal{V}^1,\ldots,\mathcal{V}^k)$-relation if there exists a chain of cycles in $\mathcal{V}^1,\ldots,\mathcal{V}^k$ which joins x and y, i.e. if $y \in \mathrm{ChLocus}_m(\mathcal{V}^1,\ldots,\mathcal{V}^k)_x$ for some m.

Remark 3.5.10. In the language of [Kol96, IV.4.8], the $\mathrm{rc}(\mathcal{V}^1,\ldots,\mathcal{V}^k)$-relation is nothing but the set-theoretic relation $\langle \mathcal{U}_1,\ldots,\mathcal{U}_k\rangle$ associated to the proper proalgebraic relation $\mathrm{Chain}(\mathcal{U}_1,\ldots,\mathcal{U}_k)$.

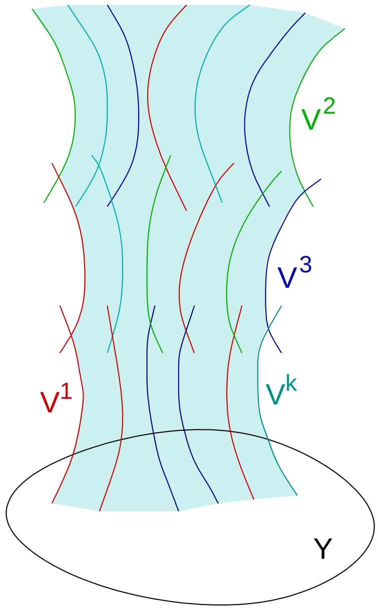


Fig. 3.2. $\mathrm{ChLocus}_3(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$

To the $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-relation we can associate a fibration, at least on an open subset of X:

Theorem 3.5.11. *[Kol96, IV.4.16] Let $\mathcal{V}^1, \ldots, \mathcal{V}^k$ be Chow families of rational curves on a normal proper variety X. Then there exist an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi : X^0 \to Z^0$ such that*

(a) the $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-relation restricts to an equivalence relation on X^0;
(b) the fibers of π are equivalence classes for the $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-relation;
(c) for every $z \in Z^0$ any two points in $\pi^{-1}(z)$ can be connected by a chain of at most $2^{\dim X - \dim Z^0} - 1$ cycles in $\mathcal{V}^1, \ldots, \mathcal{V}^k$.

Definition 3.5.12. In the above assumptions, if π is the constant map we say that X is $\mathsf{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-connected.

4

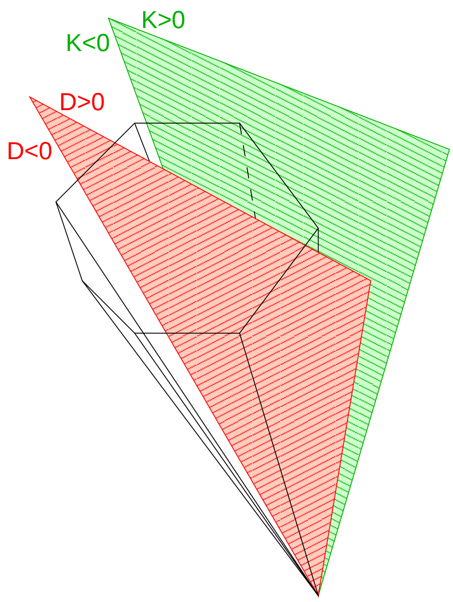
Rational curves on Fano varieties

4.1 Generalities on Fano varieties

Definition 4.1.1. A smooth complex projective variety is called Fano if its anti-canonical bundle $-K_X$ is ample.

Example 4.1.2. Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of k hypersurfaces of degrees $d_1, \ldots, d_k$. Then $K_X = \mathcal{O}(-n-1+\sum d_i)$, so X is Fano if and only if $\sum d_i < n+1$.

The definition immediately yields that the Mori cone of a Fano variety is entirely contained in the K_X-negative part of the vector space $\mathrm{N}_1(X)$, and by the Cone theorem we can conclude that the Mori cone of a Fano variety is polyhedral.



Remark 4.1.3. If X is a Fano variety and D is an effective divisor on X, then there exists at least one extremal ray R in $\mathrm{NE}(X)$ such that $D \cdot C > 0$ for every curve C whose numerical class lies in R.
In fact, if this were not the case, we would have that $D \cdot C \leq 0$ for every irreducible curve $C \subset X$, against the effectiveness of D.

A remarkable property of Fano varieties is that they are covered by rational curves of low degree, as we have proved in theorem 2.2.1.

Remark 4.1.4. As a consequence of remark 1.4.2 we have that the families $\{V^i \subset \mathrm{Ratcurves}^n(X)\}$ containing rational curves with degree $\leq n+1$ are only a finite number; so for at least one index i we have that $\mathrm{Locus}(V^i) = X$. Among these families we choose one with minimal anticanonical degree, and call it a minimal covering family. Note that by remark 3.2.7 every such family is locally unsplit.

4.2 Horizontal curves

Definition 4.2.1. Let X, Y and Z be irreducible schemes. Let $\pi : U \to Z$ be a morphism defined on a dense open subset $U \subset X$, and let $f : Y \to X$ be a morphism such that $f(Y) \cap U$ is nonempty.
A relative deformation of f over Z, parametrized by a connected pointed scheme $(S, 0)$ with a fixed base subscheme $B \subset Y$, is a morphism

$$F = \{f_s\} : Y \times S \longrightarrow X$$

which satisfies the following conditions:

(a) $f_0 = f$;
(b) $F_{|B \times S} = (f \circ pr_Y)_{|B \times S}$;
(c) $\pi \circ f_s = \pi \circ f$ for every $s \in S$.

Such deformations are parametrized by a scheme which is denoted by $\mathrm{Hom}_Z(Y, X; f_{|B})$.

If $f : C \to X$ is a curve of positive genus and π is a proper morphism, we can replace f with a morphism $f' : C \to X$ which has no relative deformations over Z:

Proposition 4.2.2. *[KMM92, Corollary 2.4] In the above setup, let $U \subset X$ be an open subset such that $\pi_{|U}$ is a proper morphism over an open subset of Z, and let $f : C \to X$ be a morphism from a curve of positive genus.*
If $\dim Z > 0$ then there exists a morphism $f' : C \to X$ such that

(a) $\pi \circ f = \pi \circ f'$;
(b) $\dim \mathrm{Hom}(C, X; f'_{|B}) = 0$ *if* B *is nonempty.*

Proof. Let us assume that $\dim \mathrm{Hom}(C, X; f_{|B}) > 0$, otherwise the statement is trivial, and let Δ be a compactification of a curve in $\mathrm{Hom}(C, X; f_{|B})$. Arguing as in theorem 2.1.1 we can prove that the induced rational map $F : S = \Delta \times C \to X$ is not defined at at least one point $(\bar{\delta}, b)$ of $\Delta \times B$.
Let $\sigma : S' \to S$ be a resolution of the indeterminacies of F, and let $\widetilde{F} = F \circ \sigma$; arguing again as in theorem 2.1.1, we can prove that the strict transform of $\Delta \times \{b\}$ contains a rational 1-cycle Z whose image in X passes through $f(b)$.

We want to prove that for every component E of Z we have that $\pi(E) = \pi(f(b))$; up to resolve further indeterminacies, we can assume that $\pi \circ \widetilde{F} : S' \to Z$ is a morphism.
We already know that $\pi \circ f_\delta = \pi \circ f$ for the generic point $\delta \in \Delta$, so for a generic point $c \in C$, if we set $\Delta_c = \sigma^*(\Delta \times \{c\})$, we have that $\pi \circ \widetilde{F}(\Delta_c)$ is a point. In particular, for every very ample divisor H on Z, the pull-back $(\pi \circ \widetilde{F})^* H$ is algebraically equivalent to a multiple of Δ_c, so it does not intersect E. It follows that $\pi(E) = \pi(f(b))$.

So if we consider the restriction of $\widetilde{F}$ to the strict transform of Δ_b, we obtain a morphism $f' : C \to X$ satisfying $\pi \circ f = \pi \circ f'$ and $H \cdot f'(C) < H \cdot f(C)$ for any ample divisor H on X. Now, if $\dim \mathrm{Hom}_Z(C, X; f'_{|B}) > 0$ we can iterate this argument, and by the bound on the degree we have to conclude after a finite number of steps. □

Lemma 4.2.3. *[KMM92, Lemma 2.5] Let* $\pi : X \dashrightarrow Z$ *be a dominant rational map between projective varieties, let* H *be an ample divisor on* X *and* D *an ample divisor on* Z*. Then there exists a constant* α *which depends only on* π, H *and* D *such that*

$$\deg f^* H \geq \alpha \deg(\pi \circ f)^* D$$

for every morphism $f : C \to X$ *from a smooth projective curve* C *whose image intersects the domain of* π.

Theorem 4.2.4. *[KMM92, Theorem 2.1] Let* X *be a Fano manifold. Suppose that there exist a nonempty open subset* U *of* X*, a smooth quasi-projective variety of positive dimension* Z *and a proper surjective morphism* $\pi : U \to Z$*. Let* z *be a general point on* Z*. Then there exists a rational curve* C *on* X *satisfying*

(a) $C \cap \pi^{-1}(z) \neq \emptyset$*;*
(b) C *is not contained in* $\pi^{-1}(z)$*;*
(c) $-K_X \cdot C \leq n+1$.

Proof. **Step 1.** *Reduction to characteristic* p.

Let us assume that we are working over a field of positive characteristic; take a general point $z \in Z$, a smooth curve $C' \subset X$ which intersects $\pi^{-1}(z)$ and a point $P_0 \in C' \cap \pi^{-1}(z)$. Let D be an ample divisor on Z and consider the constant α which appears in lemma 4.2.3 setting $H = -K_X$; let $f : C' \to C'$ be a Frobenius morphism satisfying

$$\deg(\pi \circ f)^* D > \frac{n}{\alpha} g(C').$$

By proposition 4.2.2 we can replace f with a morphism $f' : C' \to X$ such that

$$\deg(\pi \circ f')^* D = \deg(\pi \circ f)^* D$$

and f' has no relative deformations over Z with base point $P = f^{-1}(P_0)$; applying lemma 4.2.3 we have that

$$\deg f'^*(-K_X) \geq \alpha \deg(\pi \circ f')^* D > ng(C'),$$

so

$$\dim \mathrm{Hom}(C', X; f'_{|\{P\}}) \geq \deg f'^*(-K_X) - ng(C') > 0.$$

This implies that there exist absolute deformations of C' and hence, by theorem 2.1.1, there exists a rational curve in X which passes through P_0; since f' has no relative deformations over Z, its deformations induce nontrivial deformations of $\pi \circ f'$ over Z, so there exists a rational curve in X through P_0 which is mapped onto a rational curve in Z. By deforming this curve we can break it into the union of rational curves of anticanonical degree $\leq n+1$, one of which intersects $\pi^{-1}(z)$.

Step 2. *From characteristic* p *to characteristic zero.*

The proof is the same as in theorem 2.2.1. □

Remark 4.2.5. The families $\{V^i \subset \mathrm{Ratcurves}^n(X)\}$ containing the horizontal curves with degree $\leq n+1$ are only a finite number by remark 1.4.2, so for at least one index i we have that $\mathrm{Locus}(V^i)$ dominates Z^0. Among these families we choose one with minimal anticanonical degree, and call it a minimal horizontal dominating family for π.

A typical situation where these morphisms arise is the construction of rationally connected fibrations associated to families of rational curves, or more generally to a finite number of proper connected prerelations as done in [Kol96, IV.4.16].

Lemma 4.2.6. *Let X be a Fano variety, and let $\pi : X \dashrightarrow Z$ be the rationally connected fibration associated to m proper connected prerelations on X; suppose that $\dim Z > 0$ and let V be a minimal horizontal dominating family for π. Then*

(a) curves parametrized by V are numerically independent from curves contracted by π;

(b) V is locally unsplit;

(c) if x is a general point in $\mathrm{Locus}(V)$ and F is the fiber containing x, then

$$\dim(F \cap \mathrm{Locus}(V_x)) = 0.$$

Proof. (a) Since X is normal and Z is proper, the indeterminacy locus E of π in X has codimension ≥ 2 [Deb01, 1.39]. Pull back an ample divisor from Z and observe that it is zero on curves contracted by π. On the other hand it intersects nontrivially curves which are not contracted by π and are not contained in E, like curves of V, since V is dominant.

(b) If for the general $x \in \mathrm{Locus}(V)$ a curve in V_x degenerates into a reducible cycle, then at least one component of this cycle is horizontal, otherwise curves in V would be numerically equivalent to curves in the fibers. But this contradicts the minimality of V among horizontal dominating families.

We postpone the proof of (c) until section 5.2. □

Corollary 4.2.7. *Let X be a Fano variety, and let $\pi : X \dashrightarrow Z$ be the rationally connected fibration associated to m proper connected prerelations on X; let V be a minimal horizontal dominating family for π. Then*

$$\deg V \leq \dim Z + 1.$$

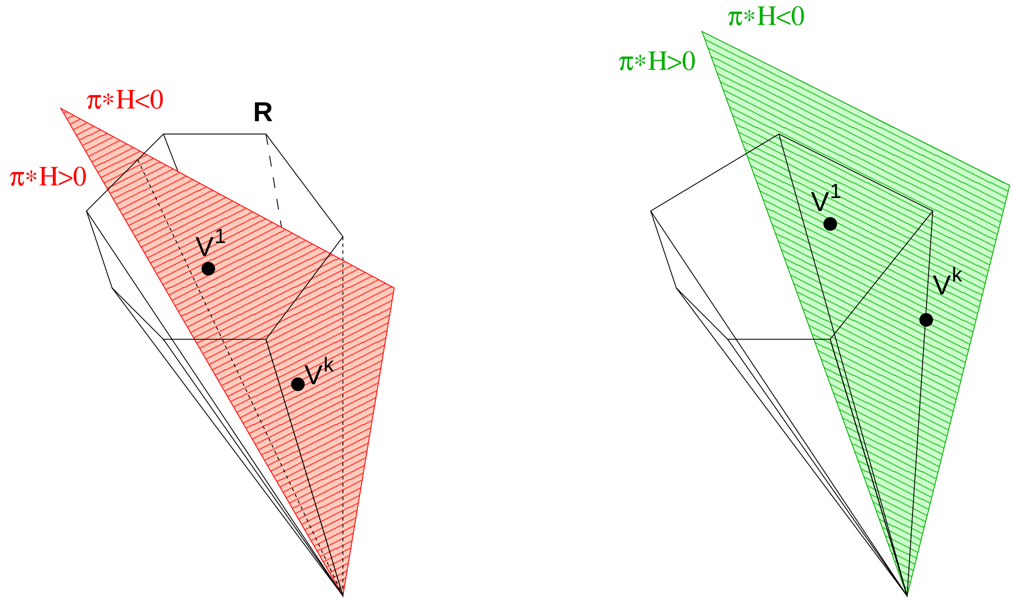
Proof. It follows from lemma 4.2.6 (c) and proposition 3.2.9. □

Lemma 4.2.8. *Let X be a Fano variety, $V^1, \ldots, V^k$ locally unsplit families of rational curves such that V^1 is covering and V^i is horizontal and dominating with respect to the rc$(\mathcal{V}^1, \ldots, \mathcal{V}^{i-1})$-fibration.*

Let $\pi : X \dashrightarrow Z$ be the rc$(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-fibration and suppose that $\dim Z > 0$.

Then either $[V^1], \ldots, [V^k]$ are contained in a proper extremal face of $\mathrm{NE}(X)$ or there exists a small extremal ray R whose exceptional locus is contained in the indeterminacy locus of π.

Proof. Since X is normal and Z is proper, the indeterminacy locus E of π in X has codimension ≥ 2 [Deb01, 1.39]; take a very ample divisor H on Z and pull it back to X; π^*H is zero on curves in $\langle [V^1], \ldots, [V^k] \rangle$, and it is positive outside the indeterminacy locus of π.
Therefore, either π^*H is nef on X and $[V^1], \ldots, [V^k]$ lie on an extremal face of $\mathrm{NE}(X)$, or π^*H is negative on an extremal ray, whose locus has to be contained in the indeterminacy locus of π and therefore has codimension greater than one in X.



□

4.3 Fano varieties of high index

Definition 4.3.1. Let X be a Fano variety of dimension n. We define the index of X as

$$r_X = \max\{m \in \mathbb{N} \mid -K_X = mL\}$$

for some (ample) divisor L on X; we also define the pseudoindex of X as

$$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m \text{ for some rational curve } C \subset X\}.$$

Remark 4.3.2. Since X is smooth, $\mathrm{Pic}(X)$ is torsion free and therefore the divisor L satisfying $-K_X = r_X L$ is uniquely determined and called the fundamental divisor of X.

It is easy to see that r_X divides i_X, and we have proved in theorem 2.2.1 that $i_X \leq n+1$; the characterization of Fano varieties of index $\geq n$ is due to Kobayashi and Ochiai:

Theorem 4.3.3. *[KO73] $r_X = n+1$ if and only if $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1))$, and $r(X) = n$ if and only if $(X, L) \simeq (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1))$.*

Fano varieties of index $n-1$, which are called del Pezzo varieties, have been classified in [Fuj90] using the Apollonius method, i.e. proving that the linear sistem $|L|$ contains a smooth divisor and constructing a ladder down to the well-known case of surfaces.
Thanks to the classification of Fano threefolds which was ruled out by Fano, Iskovskikh ([Isk77], [Isk78]), Mori and Mukai ([MM82], [Muk89]), the same method works for Fano varieties of index $n-2$, called Mukai varieties; in [Muk89] Mukai announced the classification assuming the existence of a smooth member in $|L|$, and this was proved by Mella in [Mel99].

5

Bounding the Picard number of X

In this chapter we list some conditions under which the numerical class in X of every curve lying in some subvariety $S \subset X$ is contained in a linear subspace of $\mathrm{N}_1(X)$ or in a subcone of $\mathrm{NE}(X)$. In particular we prove that if X is rationally connected with respect to k unsplit families then $\rho_X \le k$.

Notation. We write by abuse of notation

$$\mathrm{N}_1(S) = \langle [V^1], \dots, [V^k] \rangle \qquad \text{or} \qquad \mathrm{N}_1(S) = \langle [C_1], \dots, [C_k] \rangle$$

if the numerical class in X of every curve $C \subset S$ can be written as $[C] = \sum_i a_i [C_i]$, with $a_i \in \mathbb{Q}$ and $C_i \in V^i$, and similarly

$$\mathrm{NE}(S) = \langle [V^1], \dots, [V^k] \rangle \qquad \text{or} \qquad \mathrm{NE}(S) = \langle [C_1], \dots, [C_k] \rangle$$

if the numerical class in X of every curve $C \subset S$ can be written as $[C] = \sum_i a_i [C_i]$, with $a_i \in \mathbb{Q}_{\ge 0}$ and $C_i \in V^i$.

5.1 Chow families

The main idea in the proof of the following lemma was first introduced by Wiśniewski in [Wiś89], and from now on will be widely used.

Lemma 5.1.1. *Let $Y \subset X$ be a closed subset and $\mathcal{V}$ a Chow family of rational curves. Then every curve contained in $\mathrm{Locus}(\mathcal{V})_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles parametrized by $\mathcal{V}$ which intersect Y.*

Proof. Consider the restriction of diagram 3.2 to $\mathcal{V}_Y := p(i^{-1}(Y \cap \mathrm{Locus}(\mathcal{V})))$ and $\mathcal{U}_Y := p^{-1}(\mathcal{V}_Y)$:

$$\begin{tikzcd} \mathcal{U}_Y \arrow[r, "i"] \arrow[d, "p"'] & X \\ \mathcal{V}_Y & \end{tikzcd}$$

Let C be a curve in $\mathrm{Locus}(\mathcal{V})_Y$ which is not an irreducible component of a cycle parametrized by $\mathcal{V}$. Then $i^{-1}(C)$ contains an irreducible curve C' which is not contained in any fiber of p and dominates C via i. Let $B = p(C')$ and let S be the surface $p^{-1}(B)$.

Note that there exists a curve C_Y in S which dominates B and such that $i(C'_Y)$ is contained in Y: this is due to the fact that the image via i of every fiber of $p_{|S}$ meets Y.

By [Kol96, II.4.19] every curve in S is algebraically equivalent to a linear combination with rational coefficients of C'_Y and of the irreducible components of fibers of $p_{|S}$ (in [Kol96, II.4.19] take $X = S$, $Y = B$ and $Z = C'_Y$).

Thus any curve in $i(S)$, and in particular C, is algebraically - hence numerically - equivalent in $i(\mathcal{U}_Y) = \mathrm{Locus}(\mathcal{V})_Y$ (and hence in X) to a linear combination with rational coefficients of $i_*(C_Y)$ and of irreducible components of cycles parametrized by $\mathcal{V}_Y$. □

Corollary 5.1.2. *Let $Y \subset X$ be a closed subset, $\mathcal{V}^1, \ldots, \mathcal{V}^k$ Chow families of rational curves, m a positive integer.*

Then every curve contained in $\mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles parametrized by $\mathcal{V}^1, \ldots, \mathcal{V}^k$.

Proof. By remark 3.5.8 we know that

$$\mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y = \bigcup_{1 \le i(j) \le k} \mathrm{Locus}(\mathcal{V}^{i(1)}, \ldots, \mathcal{V}^{i(m)})_Y,$$

so every irreducible component of $\mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ is contained in $\mathrm{Locus}(\mathcal{V}^{i(1)}, \ldots, \mathcal{V}^{i(m)})_Y$ for some m-uple $(i(1), \ldots, i(m))$.

Then we note that the corollary is true for $\mathrm{Locus}(\mathcal{V}^{i(1)}, \ldots, \mathcal{V}^{i(m)})_Y$, applying m times lemma 5.1.1 with $Y_0 = Y$ and $Y_j = \mathrm{Locus}(\mathcal{V}^{i(1)}, \ldots, \mathcal{V}^{i(j)})_Y$ (which is closed by remark 3.5.3). □

Proposition 5.1.3. *Let $\mathcal{V}^1, \dots, \mathcal{V}^k$ be Chow families of rational curves on X and let $\pi : X^0 \to Z^0$ be the rc($\mathcal{V}^1, \dots, \mathcal{V}^k$)-fibration.*
Let $Y \subset X$ be a closed subset which dominates Z^0 via π; then every curve in X is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles in $\mathcal{V}^1, \dots, \mathcal{V}^k$.

Proof. By theorem 3.5.11, every couple of points in a general fiber of π can be connected by a chain of cycles belonging to $\mathcal{V}^1, \dots, \mathcal{V}^k$ of length at most $M = 2^{\dim X - \dim Z^0} - 1$. In particular it follows that $\mathrm{ChLocus}_M(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ is dense in X and, being closed by remarks 3.5.3 and 3.5.8, it coincides with X. Then the claim follows from corollary 5.1.2. $\square$

Corollary 5.1.4. *Suppose that X is rationally connected with respect to some Chow families $\mathcal{V}^1, \dots, \mathcal{V}^k$; then every curve in X is numerically equivalent to a linear combination with rational coefficients of the irreducible components of cycles parametrized by $\mathcal{V}^1, \dots, \mathcal{V}^k$. In particular, if X is rationally connected with respect to k quasi-unsplit families then $\rho_X \leq k$.*

Proof. We apply proposition 5.1.3 with $\pi : X \to \{*\}$ the rc($\mathcal{V}^1, \dots, \mathcal{V}^k$)- fibration and Y any point in X. The second part follows from the fact that all cycles parametrized by a quasi-unsplit family are numerically proportional. $\square$

5.2 Unsplit families

The results in the previous section can be strengthened if we consider unsplit families of rational curves instead of Chow families.

Lemma 5.2.1. *[Occ03, Lemma 1] Let $Y \subset X$ be a closed subset and V an unsplit family of rational curves. Then every curve contained in $\mathrm{Locus}(V)_Y$ is numerically equivalent to a linear combination with rational coefficients*

$$\lambda C_Y + \mu C_V,$$

where C_Y is a curve in Y, C_V belongs to the family V and $\lambda \geq 0$.

Note that the improvement with respect to lemma 5.1.1 is the claim $\lambda \geq 0$.

Proof. Let C be a curve contained in $\mathrm{Locus}(V)_Y$; if $C \subseteq Y$ or C is a curve parametrized by V we have nothing to prove, so we can suppose that this is not

the case.
In particular, keeping the notation of diagram 3.1 and setting

$$V_Y := p(i^{-1}(Y \cap \mathrm{Locus}(V))),$$

we have that $i^{-1}(C)$ contains an irreducible curve C' which is not contained in any fiber of π and dominates C via i; let $B' := \pi(C') \subset V_Y$, let $\nu : B \to B'$ be the normalization of B' and let S' be the surface $\pi^{-1}(B')$. By base change we obtain the following diagram:

$$\begin{tikzcd} S_B \arrow[r, "\bar{\nu}"] \arrow[d] & p^{-1}(V_Y) \arrow[r, "i"] \arrow[d, "\pi"] & X \\ B \arrow[r, "\nu"'] & V_Y & \end{tikzcd}$$

Let now $\mu : S \to S_B$ be the normalization of S_B; by standard arguments (see for instance [Wiś89, 1.14]) it can be shown that S is a ruled surface over the curve B; consider now the following diagram, where $j := i \circ \bar{\nu} \circ \mu$:

$$\begin{tikzcd} S \arrow[r, "j"] \arrow[d, "p"'] & X \\ B & \end{tikzcd}$$

Let f be a fiber of p and let C_Y be a curve in S which dominates B and whose image via j is contained in Y; such a curve exists since the image via j of every fiber of p meets Y.

Since S is a ruled surface, every curve in S is algebraically equivalent to a linear combination with rational coefficients of C_Y and f.
Therefore every curve in $j(S)$ is algebraically - hence numerically - equivalent in X to a linear combination with rational coefficients

$$\lambda j_*(C_Y) + \mu j_*(f),$$

where $j_*(C_Y)$ is a curve contained in Y or the zero cycle, and $j_*(f)$ is a curve of the family V.

Note that the proof actually yields that $\lambda \geq 0$; in fact, let C_S be an irreducible curve in S which dominates C via j; in S we can write $C_S \equiv \lambda C_Y + \mu f$, and intersecting with f we obtain that $\lambda \geq 0$. $\square$

Corollary 5.2.2. *Let V be a family of rational curves and x a point in X such that V_x is unsplit. Then* $\mathrm{N}_1(\mathrm{Locus}(V_x)) = \mathrm{NE}(\mathrm{Locus}(V_x)) = \langle [V] \rangle$.

Proof of lemma 4.2.6 (c). From corollary 5.2.2 we know that any curve in $\mathrm{Locus}(V_x)$ is numerically proportional to V, while proposition 5.1.3 applied to F implies that all curves in F can be written as linear combinations of curves contracted by π. □

Corollary 5.2.3. *Let R_1 be an extremal ray of X, R^1 a family of deformations of a minimal extremal curve in R_1, x a point in* $\mathrm{Locus}(R^1)$ *and V an unsplit family of rational curves, independent from R^1.*
Then $\mathrm{NE}(\mathrm{ChLocus}_m(V)_{\mathrm{Locus}(R^1_x)}) = \langle [V], [R^1] \rangle$.

Proof. Since

$$\mathrm{ChLocus}_m(V)_{\mathrm{Locus}(R^1_x)} = \mathrm{Locus}(V)_{\mathrm{ChLocus}_{m-1}(V)_{\mathrm{Locus}(R^1_x)}},$$

iterating lemma 5.2.1 m times, any curve C in $\mathrm{ChLocus}_m(V)_{\mathrm{Locus}(R^1_x)}$ can be written as

$$C \equiv \lambda C_1 + \mu C_V$$

with $C_1 \in R^1$, $C_V \in V$ and $\lambda \geq 0$; so we have only to prove that $\mu \geq 0$.
If $\mu < 0$, then we can write $C_1 \equiv \alpha C_V + \beta C$ with $\alpha, \beta \geq 0$; but since C_1 is extremal we have that both $[C]$ and $[C_V]$ belong to R_1, a contradiction. □

Remark 5.2.4. More generally, if σ is an extremal face of $\mathrm{NE}(X)$, F is a fiber of the associated contraction and V is an unsplit family independent from σ, the same proof shows that

$$\mathrm{NE}(\mathrm{Locus}(V)_F) = \langle \sigma, [V] \rangle.$$

5.3 Special Fano varieties: proof of Theorem 1

In this section we will prove

Theorem 1. *Let X be a Fano variety of dimension n and pseudoindex $i_X \geq \frac{n+3}{3}$; then conjecture B holds if*

(∗) *X has a covering unsplit family of rational curves.*

Moreover we will show that condition (∗) is fulfilled in the following cases:

- X has Picard number $\rho_X \geq 2$ and either admits a fiber type contraction or it admits no small contractions (theorem 5.3.2);
- X has dimension $n \geq 6$, Picard number $\rho_X \geq 2$ and pseudoindex $i_X = n-3$ (theorem 5.3.3).

Note that if X has Picard number 1 then the inequality in conjecture B (which is equivalent to $i_X \leq n+1$) follows immediately from theorem 2.2.1, while the characterization of $\mathbb{P}^n$ as the only case for which equality holds is contained in [CMSB02] and [Keb02], as we have already observed in section 3.2.

Notation. From now on we will denote with

R_i an extremal ray of $\mathrm{NE}(X)$,
R^i the (unsplit) family of deformations of a minimal rational extremal curve in R_i,
φ_{R_i} (or φ_i) the contraction associated to R_i.

Proof of Theorem 1. Let V be a covering unsplit family of rational curves on X, and consider the rc$V$-fibration $\pi : X^0 \to Z^0$: if $\dim Z^0 = 0$ then $\rho_X = 1$ by corollary 5.1.4 and we conclude, otherwise take a minimal horizontal dominating family V'.
From lemma 4.2.6 we know that V'_x is unsplit for general $x \in \mathrm{Locus}(V')$. Then applying lemma 3.5.5 with $Y = \mathrm{Locus}(V'_x)$ we obtain

$$\begin{aligned} 3i_X - 3 \geq n &\geq \dim \mathrm{Locus}(V)_{\mathrm{Locus}(V'_x)} \\ &\geq \dim \mathrm{Locus}(V'_x) + \deg V - 1 \\ &\geq \deg V' + \deg V - 2 \\ &\geq \deg V' + i_X - 2 \end{aligned}$$

so $\deg V' \leq 2i_X - 1$ and therefore V' is unsplit.
Take the rc(V, V')-fibration $\pi' : X' \to Z'$: if $\dim Z' = 0$ then from corollary 5.1.4 we have $\rho_X = 2$, and the inequality in conjecture B follows since we can assume that $i_X \leq \frac{n+2}{2}$ (the case $i_X > \frac{n+2}{2}$ has been dealt with in [Wiś90b]). The characterization of the equality is a consequence of the following theorem, noting that if $i_X = \frac{n+2}{2}$ then $\deg V' = i_X$ and $\dim \mathrm{Locus}(V'_x) = i_X - 1$, hence V' is covering by remark 3.2.10:

Theorem 5.3.1. *[Occ03, Theorem 1] A smooth complex variety X of dimension n is isomorphic to a product of projective spaces $\mathbb{P}^{n(1)} \times \ldots \times \mathbb{P}^{n(k)}$ if and only if* *there exist k independent unsplit covering families of rational curves $V^1, \dots, V^k$ of degrees $n(1)+1, \dots, n(k)+1$ such that*

$$X = \mathrm{Locus}(V^1, \dots, V^k)_x$$

for a general $x \in X$.

If else $\dim Z' = 0$ take a minimal dominating family V'' with respect to π'.

For general $x \in \mathrm{Locus}(V'')$, denote by F' the fiber of π' containing x: then F' is an equivalence class with respect to the $\mathrm{rc}(V, V')$-relation, so $F' \supseteq \mathrm{Locus}(V, V')_y$ for some y and theorem 3.5.4 yields

$$\dim F' \geq \deg V + \deg V' - 2 \geq 2i_X - 2.$$

By lemma 4.2.6 we know that $\dim(F' \cap \mathrm{Locus}(V''_x)) = 0$, so

$$3i_X - 3 \geq n \geq \dim F' + \dim \mathrm{Locus}(V''_x) \geq 2i_X - 2 + \deg V'' - 1,$$

that is

$$\deg V'' \leq i_X.$$

This is impossible unless $\deg V = \deg V' = \deg V'' = i_X$ and $\dim \mathrm{Locus}(V_x) = \dim \mathrm{Locus}(V'_x) = \dim \mathrm{Locus}(V''_x) = i_X - 1$; in this case remark 3.2.10 yields that all these families are covering.

Moreover, X is $\mathrm{rc}(V, V', V'')$-connected: in fact, the general fiber of the $\mathrm{rc}(V, V', V'')$-fibration contains $\mathrm{Locus}(V, V', V'')_x$ for some x, hence by theorem 3.5.4 it has dimension $\geq 3i_X - 3 \geq n$.

So we can apply theorem 5.3.1 to obtain that $X \simeq (\mathbb{P}^{i_X - 1})^3$. □

Theorem 5.3.2. *Let X be a Fano variety of dimension n, Picard number $\rho_X \geq 2$ and pseudoindex $i_X \geq \frac{n+3}{3}$. If X has a fiber type extremal contraction or it has no small contractions then X has a covering unsplit family of rational curves.*

Proof. First of all suppose that there exists a fiber type contraction $\varphi : X \to W$, and let V_φ be a minimal horizontal dominating family for φ; since every fiber type extremal contraction can be seen as the $\mathrm{rc}\mathcal{V}$-fibration with respect to some covering Chow family $\mathcal{V}$, we can apply corollary 4.2.7 and obtain that $\deg V_\varphi \leq \dim W + 1$. Moreover lemma 4.2.6 yields that the general fiber F of φ satisfies

$$\dim F \leq \dim X - \deg V_\varphi + 1 \leq 2i_X - 2.$$

By adjunction we have $K_F = (K_X)_{|F}$, so F is a Fano variety; in particular F has a minimal covering family V_F of degree $\leq \dim F + 1 \leq 2i_X - 1$.
This means that through a general point of X there passes a curve of degree $\leq 2i_X - 1$, and since the families of rational curves with bounded degree are a finite number, one of them must be covering; the bound on the degree implies that this family is also unsplit.

Suppose now that all the extremal contractions of X are divisorial and, by contradiction, that there does not exist any covering unsplit family of rational curves.
Let V be a minimal covering family on X; since we are assuming that V is not unsplit we have $\deg V \geq 2i_X$.
Consider the Chow family $\mathcal{V}$ associated to V: since $\deg V \leq n+1 \leq 3i_X - 2 < 3i_X$, reducible cycles in $\mathcal{V}$ split into exactly two irreducible components. To each one of them we associate the corresponding irreducible component of $\mathrm{Ratcurves}^n(X)$, which is an unsplit family.
We denote by $\mathcal{B}$ the finite set of pairs of families $(W^i, \overline{W}^i)$ such that:

- $[W^i]$ is numerically independent from $[\overline{W}^i]$;
- $[W^i] + [\overline{W}^i] = [V]$;
- W^i and $\overline{W}^i$ contain irreducible components of cycles of $\mathcal{V}$.

Consider now the rc$\mathcal{V}$-fibration $\pi : X^0 \to Z^0$.

Claim. $\dim Z^0 = 0$.

Suppose by contradiction that Z^0 has positive dimension, and take V' a minimal horizontal dominating family for π; we know from lemma 4.2.6 that for a general fiber F we have

$$\dim \mathrm{Locus}(V'_x) + \dim F \leq n,$$

which implies

$$\deg V' \leq n + 1 - \dim F \leq n - 2i_X + 2 < i_X,$$

a contradiction which proves the claim.

As a corollary we obtain that $\mathrm{N}_1(X)$ is generated as a vector space by the numerical classes of the irreducible components of cycles in $\mathcal{V}$ (proposition 5.1.3) so, since we are assuming $\rho_X \geq 2$, the set $\mathcal{B}$ is nonempty.
Note also that if $[V]$ is extremal in $\mathrm{NE}(X)$ then all the irreducible components of cycles in $\mathcal{V}$ are numerically proportional to $[V]$ and in this case $\rho_X = 1$, so we can assume that $[V]$ is not extremal.

Take now R_1 to be a divisorial extremal ray of X, and let E_1 be its exceptional locus. First of all we claim that $E_1 \cdot V = 0$: if this were not the case, for a general $x \in X$ the set $\mathrm{Locus}(R^1)_{\mathrm{Locus}(V_x)}$ would be nonempty, so by lemma 3.5.5 and proposition 3.2.9 we would have

$$\dim \mathrm{Locus}(R^1)_{\mathrm{Locus}(V_x)} \geq \dim \mathrm{Locus}(V_x) + \deg R^1 - 1 \geq 3i_X - 2 > n,$$

a contradiction. Since $\mathrm{N}_1(X)$ is generated by the numerical classes of the irreducible components of cycles in $\mathcal{V}$, we can find a pair $(W^1, \overline{W}^1) \in \mathcal{B}$ such that $E_1 \cdot W^1 < 0$ and $E_1 \cdot \overline{W}^1 > 0$; in particular we have that $\mathrm{Locus}(W^1) \subseteq E_1$.

Suppose that $[W^1] \neq [\lambda R^1]$ and let x be a point in $\mathrm{Locus}(W^1) \cap \mathrm{Locus}(\overline{W}^1)$.
Let C be a curve whose degree is minimum among curves in R_1 passing through x, and consider the associated proper family $R^{1\prime}$; by corollary 5.2.3, the class of every curve in $\mathrm{Locus}(R^{1\prime}, W^1)_x$ can be written as a linear combination with positive coefficients of $[R^{1\prime}]$ and $[W^1]$, so

$$\dim(\mathrm{Locus}(\overline{W}^1_x) \cap \mathrm{Locus}(R^{1\prime}, W^1)_x) = 0;$$

on the other hand, by remark 3.5.6 we have

$$\dim \mathrm{Locus}(R^{1\prime}, W^1)_x \geq 2i_X - 1,$$

and therefore

$$\dim(\mathrm{Locus}(\overline{W}^1_x) \cap \mathrm{Locus}(R^{1\prime}, W^1)_x) \geq 3i_X - 2 - n > 0.$$

We thus get a contradiction, unless $[W^1] = [\lambda R^1]$.

Note that this argument also shows that for all $i \neq 1$ we have $E_1 \cdot W^i = E_1 \cdot \overline{W}^i = 0$.

By remark 4.1.3 there exists an extremal ray R_2 on which E_1 is positive ; let E_2 be the exceptional locus of R_2.
We repeat the same argument and we find a pair $(W^2, \overline{W}^2)$ such that $[R^2] = [\mu W^2]$ and $E_2 \cdot \overline{W}^2 > 0$.
If the plane Π_1 spanned in $\mathrm{N}_1(X)$ by $[V]$ and $[R^1]$ is different from the plane Π_2 spanned by $[V]$ and $[R^2]$, then $[R^1]$, $[R^2]$ and $[\overline{W}^2]$ are independent; moreover $\mathrm{Locus}(\overline{W}^2, R^2, R^1)_x$ is nonempty for every $x \in \mathrm{Locus}(\overline{W}^2)$, so by remark 3.5.6 we get

$$\dim \mathrm{Locus}(\overline{W}^2, R^2, R^1)_x \geq 3i_X - 2 > n,$$

a contradiction.
So we can assume that $\Pi_1 = \Pi_2 := \Pi$ and we choose a basis of $\mathrm{N}_1(X)$ formed by $[R^1], [V]$ and by classes $[W^i]$, $[\overline{W}^i]$ not contained in Π.
Since the divisors E_1 and E_2 are zero on all the elements of the basis but $[R^1]$, they are proportional in $\mathrm{N}^1(X)$; but $E_1 \cdot R^1 < 0$ and $E_2 \cdot R^1 > 0$, so $E_1 = -kE_2$ with $k > 0$. One can now compute the intersection number of E_1 and E_2 with any curve which meets $E_1 \cup E_2$ without being contained in it, and this leads to a contradiction. □

Theorem 5.3.3. *Let X be a Fano variety of dimension $n \geq 6$, Picard number $\rho_X \geq 2$ and pseudoindex $i_X = n - 3$. Then X has a covering unsplit family of rational curves.*

Proof. Let V be a minimal covering family of rational curves on X.
Proposition 3.2.9 and corollary 5.2.2 yield that if $\deg V = n + 1$ then $\rho_X = 1$, contrary to the assumptions, so we can assume that $\deg V \leq n$. Moreover if $n \geq 7$ or $n = 6$ and $\deg V \leq 5$ the inequality

$$\deg V \leq n < 2(n-3) = 2i_X,$$

is satisfied, hence V is unsplit. We are thus left with the case when $n = \deg V = 6$.

Let $x \in X$ be a general point and let D be an irreducible component of $\mathrm{Locus}(V_x)$; since V is locally unsplit we have $\mathrm{N}_1(D) = \langle [V] \rangle$ by corollary 5.2.2, and by proposition 3.2.9 we know that $\dim D \geq \deg V - 1 = 5$. We are assuming $\rho_X \geq 2$ so it cannot be $D = X$, therefore D is an effective divisor.
The rc$\mathcal{V}$-fibration $\pi : X^0 \to Z^0$ has fibers of dimension ≥ 5; if Z^0 has positive dimension, take V' to be a minimal horizontal dominating family for π. Then for a general fiber F of π we have

$$\dim(F \cap \mathrm{Locus}(V'_x)) \geq 5 + \deg V' - 1 - 6 \geq i_X - 2 \geq 1,$$

contradicting lemma 4.2.6.
It follows that X is rc$\mathcal{V}$-connected; in particular, keeping the same notation as in theorem 5.3.2, we know that $\mathrm{N}_1(X)$ is generated as a vector space by the numerical class of V and the numerical classes of the families W^i such that $(W^i, \overline{W}^i) \in \mathcal{B}$ for some $\overline{W}^i$.
If V is quasi-unsplit then W^i and $\overline{W}^i$ are numerically proportional to V for every i, hence $\rho_X = 1$, a contradiction; otherwise consider the nonempty set of pairs

$(W^i, \overline{W}^i) \in \mathcal{B}$ such that $[W^i] \neq [\alpha V]$ and the (non negative) intersection number $D \cdot V$.

If $D \cdot V = 0$ then $D \cdot W^i < 0$ for some i (otherwise D would be numerically trivial), so it contains curves in W^i against the fact that $\mathrm{N}_1(D) = \langle [V] \rangle$.

If else $D \cdot V > 0$ then for every i we have that either $D \cdot W^i > 0$ or $D \cdot \overline{W}^i > 0$; but in this case either $\mathrm{Locus}(W^i_x) \cap D$ or $\mathrm{Locus}(\overline{W}^i_x) \cap D$ is nonempty. Suppose without loss of generality that we are in the first case; since W^i is noncovering we know that $\dim \mathrm{Locus}(W^i_x) \geq 2$; hence $\dim(\mathrm{Locus}(W^i_x) \cap D) \geq 1$, against the fact that $\mathrm{N}_1(\mathrm{Locus}(W^i_x)) = \langle [W^i] \rangle$ and $\mathrm{N}_1(D) = \langle [V] \rangle$. In both cases we reach a contradiction. $\square$

5.4 Proof of Theorem 3: the higher-dimensional case

We conclude this chapter with the proof of theorem 3 in case X has dimension greater or equal than 6; namely we prove the following

Theorem 3. *Let X be a Fano variety of dimension $n \geq 6$, pseudoindex $i_X = n-3$ and Picard number $\rho_X \geq 2$. Then $\mathrm{NE}(X)$ is generated by ρ_X rays.*
More precisely, we have the following list of possibilities, where F stands for a fiber type contraction, D_i for a divisorial contraction whose exceptional locus is mapped to a i-dimensional subvariety and S for a small contraction.
All cases are effective.

$\dim X$	ρ_X	R_1	R_2	R_3
6	2	F	F	
		F	D_1	
		F	D_2	
		F	S	
	3	F	F	F
7	2	F	F	
		F	D_2	
8	2	F	F	

Proof. From theorem 5.3.3 and theorem 1 we know that conjecture B holds in our assumptions. This implies that if $\dim X \geq 8$ then $\rho_X = 1$ except if $X \simeq \mathbb{P}^4 \times \mathbb{P}^4$, so we have to deal only with varieties of dimension 6 and 7. Moreover, conjecture B implies that in this case $\rho_X \leq 2$ except if $X \simeq \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, so we are left with the case when $\rho_X = 2$, and the first statement of the theorem is trivial.

We prove that at least one of the extremal contractions on X is of fiber type. Take a minimal covering unsplit family V, and assume that this is not the case; in particular $[V]$ is not extremal, so by lemma 4.2.8 there exists a small extremal ray R_1. Consider a general nontrivial fiber F_1 of φ_1 and an irreducible component D_1 of $\mathrm{Locus}(V)_{F_1}$. From remark 5.2.4 we know that $\mathrm{NE}(D_1) = \langle [V], [R^1] \rangle$; moreover the fiber locus inequality 3.2.11 yields that $\dim F_1 = n - 2$, so by lemma 3.5.5 we have that $\dim D_1 = n$, i.e. $D_1 = X$. This implies that $[V]$ is extremal in $\mathrm{NE}(X)$, a contradiction which proves that $\mathrm{NE}(X)$ has at least one fiber type ray.

Now let R_1 be the fiber type ray of $\mathrm{NE}(X)$; we observe that the general nontrivial fibers F_i of φ_i satisfy

$$\dim F_1 + \dim F_2 \leq n.$$

In case X has dimension 7, the fiber locus inequality applied to the fiber type contraction holds that $\dim F_1 \geq 3$, so F_2 must have dimension ≤ 4; the fiber locus inequality applied to φ_2 shows that φ_2 must necessarily contract a divisor to a surface.

The same argument shows that in case X has dimension 6 the contraction φ_2 cannot be of type D_0. $\square$

6

Fano fivefolds with a quasi-unsplit locally unsplit covering family

In this chapter and in the following we will deal with five-dimensional varieties, proving theorems 2 and 4 and concluding the proof of theorem 3. We divide our study into two cases: in this chapter we will deal with fivefolds which admit a quasi-unsplit locally unsplit covering family of rational curves, while in chapter 7 we will consider the case when every locally unsplit covering family on X is not quasi-unsplit.

6.1 Proof of Theorem 2

Theorem 2. *If X is a Fano fivefold which admits a quasi-unsplit locally unsplit covering family of rational curves, then conjecture B holds for X.*

Proof. Let $V \subseteq \mathrm{Ratcurves}^n(X)$ be a quasi-unsplit locally unsplit covering family for X; by remark 4.1.4 we can assume that $\deg V \leq 6$. If $\deg V = 6$ then $X = \mathrm{Locus}(V_x)$ for general $x \in X$ and $\rho_X = 1$ by lemma 5.1.1; from the discussion at the beginning of section 5.3 we know that in this case conjecture B is satisfied, therefore we can assume that $\deg V \leq 5$.
Note that if $i_X \geq 3$ then $\deg V \leq 5 < 2i_X$, so V is unsplit and the result follows from theorem 1; so from now on we will assume that $\mathbf{i_X = 2}$, and we will prove that $\rho_X \leq 5$.

Consider the rc$\mathcal{V}$-fibration $\pi : X^0 \to Z^0$: if $\dim Z^0 = 0$, since V is quasi-unsplit we have that $\rho_X = 1$ by corollary 5.1.4 and we conclude; otherwise take a minimal horizontal dominating family V'.

Case 1. V' is quasi-unsplit.

Consider the $\mathrm{rc}(\mathcal{V}, \mathcal{V}')$-fibration $\pi' : X' \to Z'$; if Z' is a point then $\rho_X = 2$ and we conclude, otherwise take a minimal horizontal dominating family V''. Recall that V'' is locally unsplit.

If V'' is not unsplit then $\deg V'' \geq 4$, so $\dim \mathrm{Locus}(V''_x) \geq 3$; moreover, since $\dim Z' \leq 3$, we have that $\mathrm{Locus}(V''_x)$ dominates Z'. Now take a general point $x \in \mathrm{Locus}(V'')$ and apply proposition 5.1.3 with $Y = \mathrm{Locus}(V''_x)$ to obtain $\rho_X = 3$.

If V'' is unsplit we can take the $\mathrm{rc}(\mathcal{V}, \mathcal{V}', V'')$-fibration $\pi'' : X'' \to Z''$: then either Z'' is a point and $\rho_X = 3$ or every minimal horizontal dominating family is unsplit. In this case we can consider the new fibration and repeat the same argument: finally we find at most five independent quasi-unsplit families on X such that X is rationally connected with respect to them, so $\rho_X \leq 5$ by corollary 5.1.4.
If there are exactly five independent families, then they must be covering and of degree 2, hence unsplit, and from theorem 5.3.1 we conclude that $X \simeq (\mathbb{P}^1)^5$.

Case 2. Every minimal horizontal dominating family V' is not quasi-unsplit.

Note that in this situation $\deg V' \geq 4$, so $\dim \mathrm{Locus}(V'_x) \geq 3$; in particular, since V' is horizontal and dominates Z^0, we have also $\dim Z^0 \geq 3$.

If $\dim Z^0 = 3$ take a general point $x \in \mathrm{Locus}(V')$, so that V'_x is unsplit. Note that $\mathrm{Locus}(V'_x)$ dominates Z^0 and $\mathrm{NE}(\mathrm{Locus}(V'_x)) = \langle [V'] \rangle$ by corollary 5.2.2, so we can apply proposition 5.1.3 to get $\rho_X = 2$.

If $\dim Z^0 = 4$ consider the $\mathrm{rc}(\mathcal{V}, \mathcal{V}')$-fibration $\pi' : X' \to Z'$.

Claim. $\dim Z' = 0$.

Assume that this is not the case and denote by F' a general fiber of π'. Then there exists a minimal horizontal dominating family V'' satisfying

$$\begin{aligned} 0 = \dim(F' \cap \mathrm{Locus}(V''_x)) &\geq \dim F' + \dim \mathrm{Locus}(V''_x) - 5 \\ &\geq 4 + \dim \mathrm{Locus}(V''_x) - 5 \\ &\geq \deg V'' - 2 \end{aligned}$$

for every $x \in F' \cap \mathrm{Locus}(V'')$. It follows that $\deg V'' = 2$ and $\dim \mathrm{Locus}(V''_x) = 1$, so V'' is unsplit and covering by remark 3.2.10. Since V'' is horizontal also with respect to the fibration π this contradicts the minimality of V', and the claim is proved.

From corollary 4.2.7 it follows that $\deg V' \leq 5$, so every reducible cycle in $\mathcal{V}'$ splits into exactly two irreducible components; moreover the family of deformations of each component is unsplit and noncovering because of the minimality of V'.
Consider the pairs $(W^i, \overline{W}^i)$ of unsplit families satisfying

- $[W^i] + [\overline{W}^i] = [V']$,
- W^i and $\overline{W}^i$ contain irreducible components of a cycle in $\mathcal{V}'$,

and let $\mathcal{B}$ be the set of these pairs.

If the numerical class of every pair in $\mathcal{B}$ lies in the plane $\Pi \subseteq \mathrm{N}_1(X)$ spanned by $[V]$ and $[V']$ then by corollary 5.1.4 we have that $\rho_X = 2$ and we conclude.
So assume by contradiction that there exists a pair $(W^1, \overline{W}^1) \in \mathcal{B}$ such that $[W^1]$, $[\overline{W}^1] \notin \Pi$, call Π' the plane spanned by $[W^1]$ and $[\overline{W}^1]$ and set

$$\mathcal{B}_{\Pi,\Pi'} = \{(W^i, \overline{W}^i) \in \mathcal{B} \mid [W^i],\ [\overline{W}^i] \in \langle [\Pi], [\Pi] \rangle \text{ and } [W^i], [\overline{W}^i] \neq [\lambda V]\}.$$

For every $(W^i, \overline{W}^i) \in \mathcal{B}_{\Pi,\Pi'}$, for every cycle $C_i + \overline{C}_i \in W^i + \overline{W}^i$ and for every point $x \in C_i$ consider

$$D_{i,x} := \mathrm{Locus}(W^i, V, \overline{W}^i)_x :$$

by remark 3.5.6, we have that $\dim D_{i,x} \geq 4$.

Remark 6.1.1. If $D \subset X$ is a divisor and V^i are families of rational curves such that V^k is noncovering, then it cannot be $\mathrm{Locus}(V^1, \ldots, V^k)_D = X$, since $\mathrm{Locus}(V^1, \ldots, V^k)_D \subseteq \mathrm{Locus}(V^k)$.

This implies that every irreducible component of $D_{i,x}$ is an effective divisor on X, which is contained in $\mathrm{Locus}(\overline{W}^i)$, and since $\overline{W}^i$ does not dominate Z^0 we have that $D_{i,x} \cdot V = 0$ for every i.
We claim that also $D_{i,x} \cdot V' = 0$: in fact, if $D_{i,x} \cdot V' > 0$ then every curve in V' intersects $\mathrm{Locus}(\overline{W}^i)$. Since V is covering we have that

$$\mathrm{Locus}(V)_{\mathrm{Locus}(V'_x)} \supseteq \mathrm{Locus}(V'_x),$$

so $\mathrm{Locus}(V, \overline{W}^i)_{\mathrm{Locus}(V'_x)} \neq \emptyset$; we apply lemma 3.5.5 and we obtain that $\dim \mathrm{Locus}(V, \overline{W}^i)_{\mathrm{Locus}(V'_x)} = 5$, against remark 6.1.1.
Obviously we can repeat the same argument with $\overline{D}_{i,x} := \mathrm{Locus}(\overline{W}^i, V, W^i)_x$ for every $x \in \overline{C}_i$, and we obtain effective divisors which are contained in $\mathrm{Locus}(W^i)$ and whose intersection with V and V' is zero.

Call T the union of all the divisors $D_{i,x}$ and $\overline{D}_{i,x}$. Now take a point $y \in X \setminus T$; since X is $\mathrm{rc}(\mathcal{V}, \mathcal{V}')$-connected, y can be joined to T by a chain of cycles in $\mathcal{V}$ and in $\mathcal{V}'$. In particular there exists a cycle Γ either in $\mathcal{V}$ or in $\mathcal{V}'$ which intersects T but is not contained in it, and since every component of T has intersection zero with V and V', it must be of the form $C_2 + \overline{C}_2$, with $(W^2, \overline{W}^2) \in \mathcal{B}$ and $[W^2], [\overline{W}^2] \notin \langle [\Pi], [\Pi'] \rangle$ (remember that every component of a reducible cycle in $\mathcal{V}$ is numerically proportional to V).
So, up to exchange W^2 and $\overline{W}^2$, there exists a component D of T such that $D \cdot W^2 > 0$; then $\mathrm{Locus}(W^2)_D$ is nonempty and by lemma 3.5.5

$$\dim \mathrm{Locus}(W^2)_D \geq \dim D + \deg W^2 - 1 \geq 5;$$

against remark 6.1.1. □

6.2 Proof of Theorem 3

Theorem 3. *Let X be a Fano fivefold of pseudoindex $i_X = 2$ and Picard number $\rho_X \geq 2$, which admits a quasi-unsplit locally unsplit covering family of rational curves. Then $\mathrm{NE}(X)$ is generated by ρ_X rays. More precisely, we have the following list of possibilities, where F stands for a fiber type contraction, D_i for a divisorial contraction whose exceptional locus is mapped to a i-dimensional subvariety and S for a small contraction. All cases are effective.*

$\dim X$	ρ_X	R_1	R_2	R_3	R_4	R_5
5	2	F	F			
		F	D_0			
		F	D_1			
		F	D_2			
		F	S			
	3	F	F	F		
		F	F	S		
		F	F	D_1		
		F	F	D_2		
		F	D_2	D_2		
	4	F	F	F	F	
		F	F	F	D_2	
	5	F	F	F	F	F

First of all, we state some technical results which will be frequently used throughout the proof.

Lemma 6.2.1. *Let V be a quasi-unsplit locally unsplit covering family of rational curves and R_1 an extremal ray of* $\mathrm{NE}(X)$ *independent from $[V]$; assume that the contraction φ_{R_1} has a three-dimensional fiber F. Then there exists a covering unsplit family which is numerically proportional to V.*

Proof. If V is not unsplit then $\deg V \geq 4$; this implies that V_x cannot be unsplit for any $x \in F$, otherwise we would have $\dim(\mathrm{Locus}(V_x) \cap F) \geq 1$ against the fact that $\mathrm{NE}(\mathrm{Locus}(V_x)) = \langle [V] \rangle$ and $\mathrm{NE}(F) = \langle [R^1] \rangle$.
Therefore through every point of $F \cap \mathrm{Locus}(V)$ there passes a reducible cycle in $\mathcal{V}$. Moreover, $\mathrm{Locus}(\mathcal{V})$ is closed since $\mathcal{V}$ is proper, and since V is covering $\mathrm{Locus}(\mathcal{V}) = X$, hence through any point of F there is a reducible cycle in $\mathcal{V}$.
It follows that F is contained in the locus of the family of deformations of one of the components of these cycles. Note that, since $\deg V \leq 5$ and V is quasi-unsplit, such a family is unsplit and numerically proportional to V.
We denote this family by αV, and by lemma 3.5.5 we know that $\dim \mathrm{Locus}(\alpha V)_F \geq 5$, so αV is covering. □

Lemma 6.2.2. *Let V be a quasi-unsplit locally unsplit covering family of rational curves and R_1 an extremal ray of* $\mathrm{NE}(X)$ *independent from $[V]$. Assume that the contraction φ_{R_1} has a three-dimensional fiber F (so we can assume that V is unsplit) and let D be an irreducible component of* $\mathrm{Locus}(V)_F$ *(note that, by lemma 3.5.5,* $\dim D \geq 4$*). Then*

(a) if either $D = X$ or $D \cdot V > 0$ then $\mathrm{NE}(X) = \langle [V], [R^1] \rangle$*;*
(b) if R_2 is a birational ray different from R_1 then $D \cdot R^2 = 0$;
(c) if R_2 is a divisorial ray and E_2 is its exceptional locus, then $E_2 \cdot V = E_2 \cdot R^1 = 0$;
(d) if R_2 is a fiber type ray then $D \cdot R^2 > 0$.

Proof. The proof of (a) is an easy consequence of corollary 5.2.3: in fact, we know from remark 5.2.4 that $\mathrm{NE}(D) = \langle [V], [R^1] \rangle$, and if D is a divisor and $D \cdot V > 0$ we can write $X = \mathrm{ChLocus}_2(V)_F$ (since V is covering), while if $D = X$ the proof is trivial.
To prove (b), we observe that the nontrivial fibers of φ_{R_2} have dimension ≥ 2, so if $D \cdot R^2 \neq 0$ then D contains a curve whose numerical class is in R_2, a contradiction.
In case (c), if either $E_2 \cdot V > 0$ or $E_2 \cdot R^1 \neq 0$ then $E_2 \cap D \neq \emptyset$. Take a point

$x \in E_2 \cap D$ and a curve C in R^2 passing through x: by (c) we know that $D \cdot C = 0$, so $C \subseteq D$, a contradiction.
Finally, to prove (d) let x be a point in D and C a curve in R^2 through x. Since C cannot be contained in D we must have $D \cdot R^2 > 0$. □

Remark 6.2.3. Note that (b) and (c) (resp. (d)) still hold if we replace R^2 with any noncovering (resp. covering) unsplit family which is independent from V and R^1.

Lemma 6.2.4. *Let R be a divisorial ray on X and E be its exceptional locus. Let R_i be the divisorial extremal rays of* $\mathrm{NE}(X)$ *different from R. Then $E \cdot R^i < 0$ for at most one index i; moreover if this happens then $E \cdot R^j = 0$ for every $j \neq i$ and* $\mathrm{NE}(E) = \langle [R], [R^i] \rangle$.

Proof. Assume that there exists an index i such that $E \cdot R^i < 0$; then we have $E = \mathrm{Locus}(R)_{\mathrm{Locus}(R^i_x)}$, so $\mathrm{NE}(E) = \langle [R], [R^i] \rangle$ by corollary 5.2.3. In particular, E cannot contain curves whose class is in R_j for $j \neq i$, so since fibers of φ_j have dimension ≥ 2 we have that $E \cdot R^j = 0$. □

We can now resume the proof of theorem 3, and classify the Mori cone of Fano fivefolds which admit a quasi-unsplit locally unsplit covering family of rational curves.

Remark 6.2.5. Note that this is the case if on X there exists a fiber type ray R: in fact, in this case, through every $x \in X$ there exists a rational curve which is contracted by φ_R and has degree ≤ 6; among the families of deformations of these curves we can choose a covering one with minimal degree, which is quasi-unsplit since R is extremal.
We will prove that this condition is also necessary, i.e. that if X has a quasi-unsplit locally unsplit covering family then at least one of the extremal rays of $\mathrm{NE}(X)$ is of fiber type.

$\boldsymbol{\rho_X = 2}$ In this case we only have to prove that $\mathrm{NE}(X)$ has a fiber type ray. Assume that this is not the case; in particular $[V]$ is not extremal, so by lemma 4.2.8 there exists a small extremal ray R_1.
Denote by R_2 the other extremal ray of $\mathrm{NE}(X)$; by lemma 6.2.1 we can assume that V is unsplit, and by lemma 6.2.2 either $\mathrm{NE}(X) = \langle [V], [R^2] \rangle$ and $[V]$ is extremal

or there exists an effective divisor D such that $D \cdot V = D \cdot R^2 = 0$, implying that D is numerically trivial on $\mathrm{NE}(X)$; in both cases we reach a contradiction.

$\boldsymbol{\rho_X = 3}$ We divide this part of the proof into three cases.

Case 1. All rays of $\mathrm{NE}(X)$ *are of fiber type.*

If two rays, say R_1 and R_2, do not lie on the same extremal face of $\mathrm{NE}(X)$, we can consider the rationally connected fibration $\pi : X \dashrightarrow Z$ associated to R^1 and R^2. Since $\rho_X = 3$ we have $\dim Z > 0$, so by lemma 4.2.8 X must have a small elementary contraction, a contradiction. Thus any two rays lie on an extremal face and therefore $\mathrm{NE}(X)$ has exactly three rays.

Case 2. In $\mathrm{NE}(X)$ *there exists a small extremal ray.*

In this case we prove that $\mathrm{NE}(X) = \langle [R^1], [R^2], [R^3] \rangle$, where R_1 is small and both R_2 and R_3 are of fiber type.
Denote the small ray by R_1, and denote by F_1 an irreducible component of a fiber of φ_{R_1}. Note that by lemma 6.2.1 we can assume that V is unsplit.

First of all we prove that X has at least one fiber type contraction: suppose that this is not the case, let $D_1 = \mathrm{Locus}(V)_{F_1}$ and apply lemma 6.2.2. Since $\rho_X = 3$ we cannot be in case (a), and so D_1 is a divisor such that $D_1 \cdot R^i = 0$ for every $i \neq 1$; as a consequence $\mathrm{NE}(X) = \langle [R^1], [R^2], [R^3] \rangle$.
If R_2 is a small ray, we can repeat the same argument with the divisor $D_2 = \mathrm{Locus}(V)_{F_2}$, and we obtain that D_2 vanishes on the face $\langle [R^1], [R^3] \rangle$; since D_1 vanishes on the face $\langle [R^2], [R^3] \rangle$ and $D_1 \cdot V = D_2 \cdot V = 0$, it must be $[R^3] = [V]$, against the assumption that X has no fiber type contractions.
So both R_2 and R_3 are divisorial. By lemma 6.2.2 (c), if we denote by E_i the exceptional locus of R_i we have that $E_i \cdot V = E_i \cdot R^1 = 0$, and we know that $E_i \cdot R^i < 0$, which implies $E_2 \cdot R^3 > 0$ and $E_3 \cdot R^2 > 0$; in particular this yields that the intersection numbers of E_2 and E_3 with every curve in X have opposite signs. The existence of curves which intersect $E_2 \cup E_3$ without being contained in it gives rise to a contradiction. We have thus proved that X has at least one fiber type contraction, associated to a ray R_2.

Suppose by contradiction that every other ray R_i of $\mathrm{NE}(X)$ is birational. By lemma 6.2.2 (b) the divisor $G := \mathrm{Locus}(R^2)_{F_1}$ satisfies $G \cdot R^i = 0$; moreover lemma 6.2.2 (a) implies that $G \cdot R^2 = 0$, so $\mathrm{NE}(X) = \langle [R^1], [R^2], [R^3] \rangle$.
The ray R_3 cannot be divisorial, otherwise we would have by lemma 6.2.2 (c) that $E_3 \cdot R^1 = E_3 \cdot R^2 = 0$, and since $E_3 \cdot R^3 < 0$ this contradicts the effectiveness of

E_3, so R_3 must be small.
Let F_3 be an irreducible component of a fiber of φ_{R_3} and let $G' := \mathrm{Locus}(R^2)_{F_3}$; by lemma 6.2.2 we have $G' \cdot R^1 = G' \cdot R^2 = 0$.
Consider a minimal horizontal dominating family V' for the fiber type contraction φ_{R_2}; then the same proof as in section 6.1 yields that V' is unsplit.
The family V' is independent either from R_1 and R_2 or from R_2 and R_3; assume without loss of generality that we are in the first case.
If V' is covering we have $X = \mathrm{Locus}(V', R^2)_{F_1} = \mathrm{Locus}(R^2, V')_{F_1}$, so $R^3 = V'$ and R_3 is of fiber type, a contradiction.
If else V' is noncovering, then by remark 6.2.3 we have $G \cdot V' = G' \cdot V' = 0$ and $[R^2] = [\lambda V']$, again a contradiction.

We have thus proved that X admits a small ray R_1 and at least two fiber type rays R_2, R_3; by lemma 6.2.1 we can assume that the families R^2, R^3 are unsplit, so by lemma 3.5.5 we have that $X = \mathrm{Locus}(R^3, R^2)_{F_1} = \mathrm{Locus}(R^2, R^3)_{F_1}$ and lemma 5.2.3 implies that $\mathrm{NE}(X) = \langle [R^1], [R^2], [R^3] \rangle$.

Case 3. In $\mathrm{NE}(X)$ *there exists at least a birational ray, but there are no small rays.*

In this case we prove that $\mathrm{NE}(X) = \langle [R^1], [R^2], [R^3] \rangle$, where at least one R_i is of fiber type, and that the possible cases are the ones listed in theorem 3.
Since X has no small contractions we know by lemma 4.2.8 that $[V]$ lies on an extremal face of $\mathrm{NE}(X)$.

Suppose that there exists a ray R_1 which does not lie in a face with $[V]$, and denote by E_1 its exceptional locus.
If either R_1 is divisorial and $E_1 \cdot V > 0$ or R_1 is of fiber type then the associated family R^1 is horizontal and dominating with respect to the rc$V$-fibration. Hence we can apply lemma 4.2.8 to V and R^1 and conclude that $[V]$ and $[R^1]$ are on the same extremal face, a contradiction.
So we can assume that R_1 is divisorial and $E_1 \cdot V = 0$. Then E_1 must be negative on another ray R_2 which lies in a face with $[V]$: in fact, E_1 cannot vanish on a face containing $[V]$, otherwise it would be ≤ 0 on the entire cone; clearly R_2 has to be divisorial. Then we can conclude from lemma 6.2.4 that in $\mathrm{NE}(X)_{E_1<0}$ there are two divisorial rays, in $\mathrm{NE}(X)_{E_1>0}$ there are only fiber type rays and $\mathrm{NE}(E_1) = \langle [R^1], [R^2] \rangle$.
Let R_3 be one of the fiber type rays; we can write $X = \mathrm{Locus}(R^3)_{E_1}$, and we have by remark 5.2.4 that $\mathrm{NE}(X) = \langle [R^1], [R^2], [R^3] \rangle$.

In the case when every extremal ray lies on a face with V we have trivially that $\mathrm{NE}(X) = \langle [V], [R^1], [R^2] \rangle$.

If X has two fiber type rays R_1, R_2 and one divisorial ray R_3, then φ_{R_3} cannot have a four-dimensional fiber F_3, otherwise we would have $X = \mathrm{Locus}(R^1)_{F_3}$ and $\rho_X = 2$, by remark 5.2.4.
Finally, in the case when X has one fiber type ray R_1 and two divisorial rays R_2 and R_3, we claim that both R_2 and R_3 have two-dimensional fibers: in fact, if R_2 has a fiber F_2 of dimension three, by lemma 6.2.1 and lemma 6.2.2 (c) we have that $E_3 \cdot V = E_3 \cdot R^2 = 0$, a contradiction.

$\boldsymbol{\rho_X = 4}$ In this case we have proved in section 6.1 that X is rationally connected with respect to four independent unsplit families V,V', V'' and V''' such that each one is horizontal with respect to the fibration associated to the previous ones.
Since the pointed locus of three among these families, say V, V' and V'', has dimension one, these families are covering by remark 3.2.10.
Moreover, if there exists a small ray R we can choose two covering families, say V and V', such that $[V], [V']$ and $[R]$ are numerically independent; then if F is a fiber of φ_R we can write $X = \mathrm{Locus}(V, V')_F$, implying that $\rho_X = 3$, a contradiction. So two cases are possible: either all rays are of fiber type or there exists a divisorial ray.

Suppose that all the rays of $\mathrm{NE}(X)$ are of fiber type. If there exist two rays R_1, R_2 which do not lie on the same extremal face of $\mathrm{NE}(X)$, we can consider the rationally connected fibration $\pi : X \dashrightarrow Z$ associated to R^1 and R^2. Since $\rho_X = 4$ we have $\dim Z > 0$, so X must have a small elementary contraction by lemma 4.2.8, a contradiction.
Then every pair of extremal rays lies on an extremal two-dimensional face of $\mathrm{NE}(X)$; it is easy to verify that in this case $\mathrm{NE}(X)$ has exactly four rays.

Suppose now that in $\mathrm{NE}(X)$ there exists a divisorial ray R.
Since X has no small contractions and $\rho_X = 4$, V, V' and V'' lie on the same extremal face σ of $\mathrm{NE}(X)$ by lemma 4.2.8, and, applying again lemma 4.2.8 to every pair of families chosen among V, V' and V'', we get that $\sigma = \langle [V], [V'], [V''] \rangle$.
Let F be a fiber of φ_R, which has dimension > 2 by proposition 3.2.11. Since $R \not\subset \sigma$ we have $\dim \mathrm{Locus}(V, V', V'')_F \geq \dim F + 3$ by lemma 3.5.5, so $\dim F = 2$, $X = \mathrm{Locus}(V, V', V'')_F$ and every curve in X can be written with positive coefficients

with respect to R and V; but V, V' and V'' play a symmetric role, so we can conclude that $\mathrm{NE}(X) = \langle [V], [V'], [V''], [R] \rangle$. $\square$

7

Fano fivefolds without a quasi-unsplit locally unsplit covering family

In this chapter we will deal with Fano fivefolds which do not have any quasi-unsplit locally unsplit covering family of rational curves. We still have to prove that

(a) conjecture B holds for these varieties,
(b) their Mori cone is one among those listed in theorem 3; more precisely, remark 6.2.5 yields that these fivefolds have no fiber type contractions, so we have to prove that the only possible cases are

$\dim X$	ρ_X	R_1	R_2
5	2	D_2	D_2
		D_2	S

Note that both these statements follow immediately from theorem 4, which we are going to prove in the next section.

7.1 Proof of Theorem 4

Theorem 4. *Let X be a Fano fivefold of pseudoindex $i_X = 2$ which does not have a covering quasi-unsplit locally unsplit family of rational curves; then $\rho_X = 2$ and X is the blow-up of $\mathbb{P}^5$ along a two-dimensional smooth quadric, or along a cubic scroll $\subset \mathbb{P}^4$, or along a Veronese surface.*

Proof. Note again that by our assumptions X has no fiber type contractions, see remark 6.2.5.

Let V be a locally unsplit covering family on X and let $\mathcal{V}$ be the associated Chow family. If $\deg V = 6$ then $X = \mathrm{Locus}(V_x)$ for a general $x \in X$, so $\rho_X = 1$ by

corollary 5.2.2 and V is quasi-unsplit, against the assumptions.
We can therefore assume that $\deg V \leq 5$.

Note that in this case, as we have observed at the beginning of section 6.1, if $i_X \geq 3$ then X admits an unsplit covering family of rational curves, so the assumption $i_X = 2$ in the statement of the theorem is not restrictive.
Note also that, since V is not quasi-unsplit, $[V]$ cannot be extremal; in particular it follows that $\rho_X \geq 2$. Moreover, since V is locally unsplit but not unsplit we have $\deg V \geq 2i_X = 4$.

Consider the pairs $(W^i, \overline{W}^i)$ of unsplit families satisfying

- $[W^i]$ is numerically independent from $[\overline{W}^i]$;
- $[W^i] + [\overline{W}^i] = [V]$,
- W^i and $\overline{W}^i$ contain irreducible components of a cycle in $\mathcal{V}$,

and let $\mathcal{B}$ be the set of these pairs.

We collect some properties of these families in the following lemmas:

Lemma 7.1.1. *The families W^i, $\overline{W}^i$ are unsplit, and moreover the only possible values for $(\dim \mathrm{Locus}(W^i), \dim \mathrm{Locus}(W^i_x))$ are $(4,2)$, $(4,3)$, $(4,4)$ or $(3,3)$.*

Proof. The families are unsplit since

$$4 = 2i_X \leq \deg W^i + \deg \overline{W}^i = \deg V \leq 5;$$

then in our assumptions they are noncovering, and the second assertion follows from proposition 3.2.9 and remark 3.2.10. ☐

Lemma 7.1.2. *If $(W^i, \overline{W}^i) \in \mathcal{B}$ and D_i and $\overline{D}_i$ are meeting components of $\mathrm{Locus}(W^i)$ and $\mathrm{Locus}(\overline{W}^i)$ we have, up to exchange D_i and $\overline{D}_i$, that $(\dim D_i, \dim \overline{D}_i)$ is either $(3,4)$ or $(4,4)$.*

Proof. By lemma 7.1.1 we have that $\dim D_i$, $\dim \overline{D}_i \geq 3$, and equality holds if and only if $D_i = \mathrm{Locus}(W^i_x)$ and $\overline{D}_i = \mathrm{Locus}(\overline{W}^i_y)$ for some x, y.
So if $\dim D_i = \dim \overline{D}_i = 3$ we have that $\dim(D_i \cap \overline{D}_i) \geq 1$, against the fact that $\mathrm{N}_1(D_i) = \langle [W^i] \rangle$ and $\mathrm{N}_1(\overline{D}_i) = \langle [\overline{W}^i] \rangle$ by corollary 5.2.2. ☐

Lemma 7.1.3. *Let R_1 be a divisorial ray of X and E_1 its exceptional locus. If there exists a pair $(W^i, \overline{W}^i) \in \mathcal{B}$ such that $E_1 \cdot W^i < 0$ and $E_1 \cdot \overline{W}^i > 0$, then $[W^i] \in R_1$.*

Proof. Since $E_1 \cdot W^i < 0$ we have that $\mathrm{Locus}(W^i) \subseteq E_1$; suppose by contradiction that $[W^i] \notin R_1$.
If $\dim \mathrm{Locus}(W^i_x) \geq 3$ for some $x \in \mathrm{Locus}(W^i)$, then lemma 3.5.5 yields that $\dim \mathrm{Locus}(W^i, R^1)_x \geq 5$, a contradiction by remark 6.1.1.
It follows that $\dim \mathrm{Locus}(W^i_x) = 2$ and so $\dim \mathrm{Locus}(W^i) = 4$ by lemma 7.1.1, hence $E_1 = \mathrm{Locus}(W^i) = \mathrm{Locus}(R^1, W^i)_x$ for some x; in particular corollary 5.2.3 implies that

$$\mathrm{NE}(E_1) = \langle [R^1], [W^i] \rangle \subset \mathrm{NE}(X)_{E_1 < 0}.$$

On the other hand, since $\dim \mathrm{Locus}(\overline{W}^i_x) \geq 2$ and $E_1 \cdot \overline{W}^i > 0$, we have that E_1 contains curves which are numerically proportional to $\overline{W}^i$, a contradiction. □

Now we resume the proof of theorem 4, and we divide it into steps:

Step 1. $\deg V = 4$

Assume by contradiction that $\deg V = 5$.
Let $x \in X$ be a general point and let D be an irreducible component of $\mathrm{Locus}(V_x)$; since V is locally unsplit we have $\mathrm{N}_1(D) = \langle [V] \rangle$ by corollary 5.2.2 and $\dim D \geq \deg V - 1 \geq 4$ by proposition 3.2.9. We are assuming $\rho_X \geq 2$, so it cannot be $D = X$, therefore D is an effective divisor.
Thus the rc$\mathcal{V}$-fibration $\pi : X^0 \to Z^0$ has fibers of dimension ≥ 4; if Z^0 has positive dimension, take V' to be a minimal horizontal dominating family for π. By lemma 4.2.6 we know that $\dim \mathrm{Locus}(V'_x) = 1$, so V' has degree 2 (hence it is unsplit) and it is covering, a contradiction.
This implies that X is rc$\mathcal{V}$-connected; in particular $\mathrm{N}_1(X)$ is generated as a vector space by the numerical class of V and the numerical classes of the families W^i such that $(W^i, \overline{W}^i) \in \mathcal{B}$ for some $\overline{W}^i$.
Consider the nonempty set of pairs $(W^i, \overline{W}^i) \in \mathcal{B}$ such that $[W^i] \neq [\alpha V]$, and the (non negative) intersection number $D \cdot V$.
If $D \cdot V = 0$ then $D \cdot W^i < 0$ for some i and so it contains curves in W^i, against the fact that $\mathrm{N}_1(D) = \langle [V] \rangle$.
If else $D \cdot V > 0$ then for every i either $D \cdot W^i > 0$ or $D \cdot \overline{W}^i > 0$; but in this case either $\mathrm{Locus}(W^i_x) \cap D$ or $\mathrm{Locus}(\overline{W}^i_x) \cap D$ is nonempty.
Suppose without loss of generality that we are in the first case; since by lemma 7.1.1 $\dim \mathrm{Locus}(W^i_x) \geq 2$, then $\dim(\mathrm{Locus}(W^i_x) \cap D) \geq 1$, against the fact that $\mathrm{N}_1(D) = \langle [V] \rangle$ and $\mathrm{N}_1(\mathrm{Locus}(W^i_x)) = \langle [W^i] \rangle$.

As a corollary we get that V is the unique locally unsplit dominating family for X up to numerical equivalence: in fact, if V' were another locally unsplit dominating family, for the general point $x \in X$ we would have $\dim(\mathrm{Locus}(V_x) \cap \mathrm{Locus}(V'_x)) \geq 1$ and so, since $\mathrm{N}_1(\mathrm{Locus}(V_x)) = \langle [V] \rangle$ and $\mathrm{N}_1(\mathrm{Locus}(V'_x)) = \langle [V'] \rangle$, the families would be proportional. But we have proved that $\deg V = \deg V' = 4$, so $[V] = [V']$.

Step 2. X is rc$\mathcal{V}$-connected

Consider the rc$\mathcal{V}$-fibration $\pi : X \dashrightarrow Z$, and assume by contradiction that $\dim Z > 0$.

Step 2a. There exists an unsplit noncovering family V' such that X is rc$(\mathcal{V}, V')$-connected.

Choose V' to be a minimal horizontal dominating family for π; we know from lemma 4.2.6 that

$$\dim \mathrm{Locus}(V'_x) + \dim F \leq 5,$$

but step 1 implies that $\dim F \geq \deg V - 1 = 3$, so

$$\deg V' - 1 \leq \dim \mathrm{Locus}(V'_x) \leq 2.$$

On the other hand, since in our assumptions V' cannot be covering and unsplit, remark 3.2.10 implies that

$$\dim \mathrm{Locus}(V'_x) \geq \deg V' \geq 2;$$

it follows that $\dim F = 3$, $\dim \mathrm{Locus}(V'_x) = 2$ and $\deg V' = 2$, so V' is unsplit and $\dim \mathrm{Locus}(V') = 4$. Moreover we know that $\mathrm{Locus}(V'_x)$ meets the general fiber of π, so X is rc$(\mathcal{V}, V')$-connected.

Step 2b. $\rho_X = 2$.

Let $\Pi \subseteq \mathrm{N}_1(X)$ be the plane spanned by $[V]$ and $[V']$ and let

$$\mathcal{B}_\Pi = \{(W^i, \overline{W}^i) \in \mathcal{B} \mid [W^i] \text{ and } [\overline{W}^i] \in \Pi\}.$$

If $\mathcal{B}_\Pi = \mathcal{B}$ then $\rho_X = 2$ by corollary 5.1.4 and we conclude, otherwise let D' be an irreducible component of $\mathrm{Locus}(V')$.

Since D' does not contain the general fiber F of π and the general F coincides with $\mathrm{Locus}(V_x)$ for some x, there exists a curve of V which intersects D' but is not entirely contained in it; therefore $D' \cdot V > 0$.

Let $V'_{D'}$ be the closed subfamily of V' such that $\mathrm{Locus}(V'_{D'}) = D'$; by lemma 3.5.5

we know that $\dim \mathrm{Locus}(V'_{D'})_{D'\cap F} \geq 4$, i.e. $\mathrm{Locus}(V'_{D'})_{D'\cap F} = D'$. Moreover we know that $\mathrm{N}_1(D' \cap F) = \langle [V] \rangle$, so lemma 5.1.1 implies that $\mathrm{N}_1(D') = \langle [V], [V'] \rangle$.
Let $(W^1, \overline{W}^1)$ be a pair in $\mathcal{B} \setminus \mathcal{B}_{\Pi}$; since $D' \cdot V > 0$ we have that either $D' \cdot W^1 > 0$ or $D' \cdot \overline{W}^1 > 0$, so we can assume without loss of generality that $\mathrm{Locus}(W^1)_{D'} \neq \emptyset$; but in this case lemma 3.5.5 yields that $\dim \mathrm{Locus}(W^1)_{D'} \geq 5$, and therefore that W^1 is covering, a contradiction since it is also unsplit by lemma 7.1.1.

Step 2c. *If R_1 is a small ray of* $\mathrm{NE}(X)$*, then $[R^1] = [W^i]$ for some i.*

Consider a general fiber F_1 of the contraction associated to R_1: since X is $\mathrm{rc}(\mathcal{V}, V')$-connected we know that through every point of F_1 there passes a curve in V, in W^i or in V'.
If V_y is unsplit for some $y \in F_1$ then $\dim(\mathrm{Locus}(V_y) \cap F_1) \geq 1$, against the fact that $\mathrm{N}_1(V_y) = \langle [V] \rangle$, $\mathrm{N}_1(F_1) = \langle [R^1] \rangle$ and $[V]$ is not extremal.
Therefore through every $y \in F_1$ there passes either a curve in V' or a reducible cycle of $\mathcal{V}$; it follows that either $F_1 \subset \mathrm{Locus}(V')$ or we can assume without loss of generality that $F_1 \subseteq \mathrm{Locus}(W^i)$ for some i.
In the first case, if R^1 is independent from V' we know from lemma 3.5.5 that $\dim \mathrm{Locus}(V')_{F_1} \geq 5$, a contradiction; so $[V'] = [R^1]$, since they both have degree 2, but this contradicts the fact that $\dim \mathrm{Locus}(R^1_x) = 3$ and $\dim \mathrm{Locus}(V'_x) = 2$.
If else $F_1 \subseteq \mathrm{Locus}(W^i)$ and R^1 is independent from W^i we reach again a contradiction by lemma 3.5.5, so $[W^i] = [\alpha R^1]$. By step 1 we know that $\deg W^i = 2$, and since R^1 has minimal degree in the ray we have also $\deg R^1 = 2$, i.e. $[R^1] = [W^i]$.

Now, in our assumptions $[V]$ is not extremal, so lemma 4.2.8 yields that one of the rays of X is small, and by step 2c it corresponds to a family W^i.
Let D_i and $\overline{D}_i$ be meeting components of $\mathrm{Locus}(W^i)$ and $\mathrm{Locus}(\overline{W}^i)$; we know that $\dim D_i = 3$, so $\dim \overline{D}_i = 4$ by lemma 7.1.2 and this implies that $\overline{D}_i$ is not contained in the indeterminacy locus of π. Let X^0 be the open subset where π is defined, take a point $x \in \overline{D}_i \cap X^0$ and consider $\mathrm{Locus}(\overline{W}^i_x)$; since $\dim Z \leq 2$ we have that either $\mathrm{Locus}(\overline{W}^i_x)$ is contained in a fiber of π or $\mathrm{Locus}(\overline{W}^i_x)$ dominates Z.
In the first case, if H is any ample divisor on Z we have that

$$(\pi^* H) \cdot V = (\pi^* H) \cdot \overline{W}^i = 0,$$

and since $\rho_X = 2$ this means that $\pi^* H$ is numerically trivial on X, a contradiction.
In the second case, every curve $\overline{C}_i \in \overline{W}^i$ which is contained in $\mathrm{Locus}(\overline{W}^i_x)$ meets

D_i, hence is part of a reducible cycle $C_i + \overline{C}_i \in \mathcal{V}$ and so X is rc$\mathcal{V}$-connected, again a contradiction.

Step 3. $\rho_X = 2$

By step 2 and corollary 5.1.4 we know that $\mathrm{N}_1(X)$ is generated as a vector space by the numerical classes of the irreducible components of cycles in $\mathcal{V}$.
Assume by contradiction that in $\mathcal{B}$ there exist two pairs $(W^1, \overline{W}^1)$, $(W^2, \overline{W}^2)$ whose classes generate a three-dimensional vector space inside $\mathrm{N}_1(X)$.

Case 3a. X has a small ray R_1.
Since we have proved that X is rc$\mathcal{V}$-connected, we can argue as in step 2c above and assume that $[R^1] = [W^1]$; let D_1 be an irreducible component of $\mathrm{Locus}(W^1)$ whose intersection with $\mathrm{Locus}(\overline{W}^1)$ is nonempty.
Let $\Pi \subset \mathrm{N}_1(X)$ be the plane generated by $[W^1]$ and $[\overline{W}^1]$ and let

$$\mathcal{B}_\Pi = \{(W^i, \overline{W}^i) \in \mathcal{B} \mid [W^i] \text{ and } [\overline{W}^i] \in \Pi\}.$$

Since $\dim D_1 = 3$, D_1 is a component of $\mathrm{Locus}(W^1_x)$ by proposition 3.2.9, hence $\mathrm{N}_1(D_1) = \langle [W^1] \rangle$ by corollary 5.2.2; moreover $\dim \mathrm{Locus}(\overline{W}^1)_{\mathrm{Locus}(W^1_x)} \geq 4$ by lemma 3.5.5.
Let $\overline{G}_1$ be a component of $\mathrm{Locus}(\overline{W}^1)_{\mathrm{Locus}(W^1_x)}$; by remark 6.1.1 $\overline{G}_1$ is an effective divisor in X, and we know by lemma 5.2.1 that $\mathrm{N}_1(\overline{G}_1) = \langle [W^1], [\overline{W}^1] \rangle$.

Suppose that $\overline{G}_1 \cdot V > 0$; then up to exchange W^2 and $\overline{W}^2$ we can assume that $\overline{G}_1 \cdot W^2 > 0$, hence $\mathrm{Locus}(W^2)_{\overline{G}_1}$ is nonempty and has dimension 5 by lemma 3.5.5, a contradiction.

So it must be $\overline{G}_1 \cdot V = 0$. In this case, $\overline{G}_1$ must be negative on W^i (up to exchange W^i and $\overline{W}^i$) for some i, so it contains curves in W^i; but since $\mathrm{N}_1(\overline{G}_1) = \langle [W^1], [\overline{W}^1] \rangle$ we have that the class of W^i must belong to Π. This implies that $\overline{G}_1$ is not numerically trivial on Π, so $\Pi \cap \overline{G}_1{}^\perp = \langle [V] \rangle$; in particular, for every pair $(W^i, \overline{W}^i) \in \mathcal{B}_\Pi$ we have that $(\overline{G}_1 \cdot W^i)(\overline{G}_1 \cdot \overline{W}^i) < 0$.
Moreover, if $\overline{G}_1 \cdot W^i < 0$ then $\dim \mathrm{Locus}(W^i) = 4$ hence $\overline{G}_1 = \mathrm{Locus}(W^i)$: in fact, if $\dim \mathrm{Locus}(W^i) = 3$ we could apply lemma 3.5.5 and obtain that $\dim \mathrm{Locus}(\overline{W}^1)_{\mathrm{Locus}(W^i_x)} = 5$, a contradiction.

We claim that $\mathcal{B}_\Pi = \{(W^1, \overline{W}^1)\}$. In fact, if there exists another pair $(W^i, \overline{W}^i) \in \mathcal{B}_\Pi$ then we can assume that $\overline{G}_1 \cdot W^i < 0$ and $\overline{G}_1 = \mathrm{Locus}(\overline{W}^1) = \mathrm{Locus}(W^i)$. In this case we can write

$$\overline{G}_1 = \mathrm{Locus}(\overline{W}^1, W^i)_x = \mathrm{Locus}(W^i, \overline{W}^1)_x$$

for some $x \in X$, hence $\mathrm{NE}(\overline{G}_1) = \langle [\overline{W}^1], [W^i] \rangle$. This means that $\overline{G}_1$ is negative on every curve it contains, against the fact that $\overline{G}_1$ contains curves in W^1 and $\overline{G}_1 \cdot W^1 > 0$.

Let $T = \mathrm{Locus}(W^1, \overline{W}^1)$, take a point $z \in T$ and a point $x \notin T$. Since X is rc$\mathcal{V}$-connected, we can join x to z with a chain of cycles in $\mathcal{V}$; let Γ be the first irreducible curve in the chain which meets T.

First of all we note that Γ cannot meet $\overline{G}_1$; in fact, since $\overline{G}_1 \cdot V = 0$ and Γ is not contained in T, Γ would be a curve in a family W^i independent from W^1 and $\overline{W}^1$, but in this case we would have $\dim \mathrm{Locus}(W^i)_{\overline{D}_1} = 5$ by lemma 3.5.5, a contradiction.

Therefore we can assume that Γ meets a component of $\mathrm{Locus}(W^1)$, and since all components of $\mathrm{Locus}(W^1)$ have dimension 3 we can assume that Γ meets D_1.

By construction, Γ cannot belong to a family W^i whose class is contained in Π and is not proportional to V; on the other hand, if Γ belongs to a family W^i whose class is not in Π, then $\dim \mathrm{Locus}(W^i, \overline{W}^i)_{D_1} = 5$ by lemma 3.5.5, a contradiction.

It follows that either Γ belongs to an unsplit family αV whose numerical class is proportional to V or Γ belongs to V.

In the first case, $\mathrm{Locus}(\alpha V)_{D_1}$ is a divisor D' such that $\mathrm{N}_1(D') = \langle [W^1], [\alpha V] \rangle$; if $D' \cdot V > 0$ then, up to exchange W^2 and $\overline{W}^2$, we can assume that $\mathrm{Locus}(W^2)_{D'}$ is nonempty and so $\dim \mathrm{Locus}(W^2)_{D'} = 5$, a contradiction.

Therefore $D' \cdot V = 0$; but since D' meets D_1 and $D' \not\supset D_1$ then $D' \cdot W^1 > 0$ and $D' \cdot \overline{W}^1 < 0$, so $D' = \overline{D}_1$ and the curve Γ is contained in T, a contradiction.

Finally, if Γ belongs to V we use the following

Lemma 7.1.4. *Let C be an irreducible curve in V. Then either $C \subset \mathrm{Locus}(V_x)$ for some x such that V_x is unsplit or $C \subset \mathrm{Locus}(W^i)$ for some unsplit family W^i such that $(W^i, \overline{W}^i) \in \mathcal{B}$.*

Proof. If there exists a point $x \in C$ such that V_x is unsplit, then we are in the first case. Otherwise, for every $x \in C$ there passes a reducible cycle $C_x^i + \overline{C}_x^i \in \mathcal{V}$. Since the families such that $[W^i] + [\overline{W}^i] = [V]$ are only a finite number, it follows that $C \subset \mathrm{Locus}(W^i)$ for some i. □

We thus have two possibilities for Γ:

(a) $\Gamma \subset \mathrm{Locus}(V_x)$ with V_x unsplit: in this case $\mathrm{Locus}(V_x) \cap D_1 \neq \emptyset$ and therefore $\dim \mathrm{Locus}(V_x) \cap D_1 \geq 1$, against the fact that $\mathrm{N}_1(\mathrm{Locus}(V_x)) = \langle [V] \rangle$ and $\mathrm{N}_1(D_1) = \langle [W^1] \rangle$;

(b) $\Gamma \subset \mathrm{Locus}(W^i)$ with $[W^i] \notin \Pi$; in this case $\mathrm{Locus}(W^i, \overline{W}^i)_{D_1}$ is nonempty and has dimension 5 by lemma 3.5.5, a contradiction.

Thus we have proved that if X has a small contraction then $\rho_X = 2$.

Case 3b. X has only divisorial contractions.
Let E_1 be the exceptional locus of a ray R_1 of $\mathrm{NE}(X)$ and consider the intersection number $E_1 \cdot V$.

If $E_1 \cdot V > 0$, for a general point $x \in X$ we have $\dim \mathrm{Locus}(R^1)_{\mathrm{Locus}(V_x)} = 4$, so that $E_1 = \mathrm{Locus}(R^1)_{\mathrm{Locus}(V_x)}$ and $\mathrm{N}_1(E_1) = \langle [R^1], [V] \rangle$ by lemma 5.2.1.
If every pair $(W^i, \overline{W}^i) \in \mathcal{B}$ lies in the plane spanned by $[V]$ and $[R^1]$ then $\rho_X = 2$ and we conclude, otherwise let $(W^i, \overline{W}^i)$ be a pair not lying in that plane: then either $E_1 \cdot W^i > 0$ or $E_1 \cdot \overline{W}^i > 0$, implying that either $E_1 \cap \mathrm{Locus}(W^i_x)$ or $E_1 \cap \mathrm{Locus}(\overline{W}^i_x)$ is nonempty and so has dimension ≥ 1, a contradiction.

If else $E_j \cdot V = 0$ for every j, then for every j there exists an index i such that $E_j \cdot W^i < 0$ and $E_j \cdot \overline{W}^i > 0$; so by lemma 7.1.3 we know that $[W^i] \in R_j$ and $E_j = \mathrm{Locus}(W^i)$.
By lemma 4.1.3, E_j is positive on an extremal ray R_k; since $E_k = \mathrm{Locus}(\overline{W}^l)$ for some l, we have that $E_j \cdot W^l < 0$, so $l = i$. Up to rename the indexes, we have that $E_j = \mathrm{Locus}(W^j)$ and $\overline{E}_j := E_k = \mathrm{Locus}(\overline{W}^j)$.

Let $T = E_j \cup \overline{E}_j$, let z be a point in T and let x be a general point of X. Since X is rc$\mathcal{V}$-connected there exists a chain of cycles in $\mathcal{V}$ connecting x and z; let Γ be the first irreducible component of one of these chains which meets T.
The curve Γ cannot be numerically proportional to V, since we are assuming $E_j \cdot V = \overline{E}_j \cdot V = 0$.
Moreover, the plane spanned by $[W^j]$ and $[\overline{W}^j]$ does not contain the class of any other family in a pair $(W^i, \overline{W}^i) \in \mathcal{B}$: in fact, if this were not the case E_j would be negative on W^i (or $\overline{W}^i$), yielding that $\mathrm{NE}(E_j) = \langle [W^j], [W^i] \rangle$, but this contradicts the fact that $E_j \cdot \overline{W}^j > 0$ so E_j contains curves in $\overline{W}^j$.
This means that Γ belongs to an unsplit family W^i which is independent from W^j and $\overline{W}^j$; then either $E_j \cdot W^i > 0$ or $\overline{E}_j \cdot W^i > 0$, which implies that either

$E_j \cdot \overline{W}^i < 0$ or $\overline{E}_j \cdot \overline{W}^i < 0$. Again this implies that $\mathrm{NE}(E_j) = \langle [W^j], [\overline{W}^i] \rangle$, contradicting the fact that $E_j \cdot \overline{W}^j > 0$.

Step 4. X has a divisorial contraction.

Suppose that both R_1 and R_2 correspond to small contractions; by step 2 we know that there exist unsplit families W^1 and W^2 such that $(R^1, W^1), (R^2, W^2) \in \mathcal{B}$, so in particular $[V] = [R^1] + [W^1] = [R^2] + [W^2]$.
Take two points $x_i \in \mathrm{Locus}(R^i)$ for $i = 1, 2$, and two irreducible components D_i of $\mathrm{Locus}(R^i, W^i)_{x_i}$; then lemma 3.5.5 implies that $\dim D_i = 4$, and by lemma 5.2.3 we know that $\mathrm{NE}(D_i) = \langle [R^i], [W^i] \rangle$.
It follows that $D_1 \cdot R^2 = D_2 \cdot R^1 = 0$; moreover, since D_i is an effective divisor for every i, we have $D_i \cdot R^i > 0$, so D_i is nef.
Write $-K_X = aD_1 + bD_2$; we have

$$\begin{aligned} 4 &= -K_X \cdot (W^1 + W^2) = \\ &= aD_1 \cdot (W^1 + W^2 + R^2) + bD_2 \cdot (W^1 + W^2 + R^1) = \\ &= aD_1 \cdot W^1 + aD_1 \cdot V + bD_2 \cdot W^2 + bD_2 \cdot V = \\ &= aD_1 \cdot W^1 + bD_2 \cdot W^2 - K_X \cdot V. \end{aligned}$$

Hence $aD_1 \cdot W^1 = bD_2 \cdot W^2 = 0$, a contradiction.

Step 5. X is the blow-up of $\mathbb{P}^5$ along a smooth surface.

We know by step 4 that X has a divisorial ray R_1; the other ray R_2 can be either small or divisorial.

Let us start assuming that R_2 is small; denote by E_1 the exceptional locus of R_1 and by G_2 a component of the exceptional locus of R_2.
The divisor E_1 is positive on R^2; it follows that $\mathrm{Locus}(R^1)_{G_2}$ is nonempty and has dimension four, so that $E_1 = \mathrm{Locus}(R^1)_{G_2}$; in particular every fiber of R_1 meets G_2 and so it is two-dimensional.
We can thus apply [AO02, Theorem 5.1] and we get that $\varphi_1 : X \to Y$ is a blow-down with center a smooth surface S .

Arguing as in step 2c we can prove that there exists a pair $(W^1, \overline{W}^1) \in \mathcal{B}$ such that $[\overline{W}^1] = [R^2]$. Take D_1 to be an irreducible component of $\mathrm{Locus}(W^1)$ which intersects $\mathrm{Locus}(\overline{W}^1)$; since $D_1 \cdot \overline{W}^1 > 0$ we have that $D_1 = \mathrm{Locus}(\overline{W}^1, W^1)_x$ and so $\mathrm{NE}(D_1) = \langle [W^1], [\overline{W}^1] \rangle$.
We claim that $W^1 = R^1$: if this is not the case then $D_1 \neq E_1$, so $\varphi_1(D_1)$ is an

effective divisor on Y. Moreover $\varphi_1(D_1)$ is ample since $\rho_Y = 1$, hence it meets S and $D_1 \cap E_1 \neq \emptyset$.
It follows that $\dim(D_1 \cap \mathrm{Locus}(R^1_x)) \geq 1$ and D_1 contains curves numerically proportional to R^1, a contradiction.

Now we show that $E_1 \cdot V > 0$.
Suppose by contradiction that $E_1 \cdot V = 0$; in this case $\mathcal{B}$ contains only the pair $(W^1, \overline{W}^1)$ and possibly a pair $(W^2, \overline{W}^2)$ with $[W^2] = [\overline{W}^2] = \frac{1}{2}[V]$ by lemma 7.1.3.
Let $T = E_1 \cup \mathrm{Locus}(\overline{W}^1)$ and take a point x outside T; since X is rc$\mathcal{V}$-connected we can join x and T with a chain of cycles in $\mathcal{V}$. Let Γ be the first irreducible component which meets T.
Clearly Γ cannot belong to $[W^1]$ and $[\overline{W}^1]$ because it is not contained in T, so it belongs either to V or to W^2. Since $E_1 \cdot V = E_1 \cdot W^2 = 0$, Γ must intersect T in points of $T \setminus E_1$.
Let y be a point in $\Gamma \cap T$ and let G_y be the irreducible component of T which contains y; by [ACO04, Lemma 9.1] we have that either $\Gamma \subset \mathrm{Locus}(V_z)$ for some z such that V_z is unsplit or $\Gamma \subset \mathrm{Locus}(W^2)$.
In the first case we have $\dim(\mathrm{Locus}(V_z) \cap G_y) \geq 1$, against the fact that $\mathrm{N}_1(V_z) = \langle [V] \rangle$ and $\mathrm{N}_1(G_y) = \langle [\overline{W}^1] \rangle$.
In the second case we consider $H = \mathrm{Locus}(W^2)_{G_y}$: by lemma 3.5.5 we have $\dim H = 4$, and by lemma 5.2.3 we have $\mathrm{NE}(H) = \langle [\overline{W}^1], [W^2] \rangle$; this implies that $H \cdot R^1 = 0$.
The image $\varphi_1(H)$ of H in Y is an effective, hence ample, divisor; therefore $\varphi_1(H) \cap S \neq \emptyset$ and $H \cap E_1 \neq \emptyset$.
For every point $t \in H \cap E_1$ we have that both $\mathrm{Locus}(W^2_t)$ and $\mathrm{Locus}(W^1_t)$ are contained in $H \cap E_1$, since $H \cdot W^1 = E_1 \cdot W^2 = 0$.
Therefore $\mathrm{Locus}(W^2, W^1)_t \subseteq H \cap E_1$, and we reach a contradiction since $\dim \mathrm{Locus}(W^2, W^1)_t = 4$ by lemma 3.5.5.

Finally, let V_Y be a minimal dominating family for Y and let V_Y^* be the family of deformations of the strict transform of a general curve in V_Y. We have

$$6 = \dim Y + 1 \geq -\varphi_1^* K_Y \cdot V_Y^* = -K_X \cdot V_Y^* + 2E_1 \cdot V_Y^*. \tag{7.1}$$

We claim that either V_Y^* is a locally unsplit dominating family for X or $\deg V_Y^* = 6$: in fact, if $\deg V_Y^* \leq 5$ and V_Y^* were not locally unsplit, there would exist a covering family of degree ≤ 3, hence unsplit, a contradiction.
If V_Y^* is locally unsplit then $[V_Y^*] = [V]$ by the final part of step one, so $E_1 \cdot V_Y^* > 0$.

We thus have that $-\varphi_1^* K_Y \cdot V_Y^* = -K_Y \cdot V_Y = 6$, and the same holds if $\deg V_Y^* = 6$; so we can conclude that Y has a minimal (hence locally unsplit by lemma 4.2.6) dominating family of degree $\dim Y + 1$; by the proof of [Keb02, Theorem 1.1] we have $Y \simeq \mathbb{P}^5$. (Note that the assumptions of the quoted result are different, but the proof actually works in our case, since for a very general y the pointed family V_{Y_y} has the properties 1-3 in [Keb02, Theorem 2.1]).

Assume now that both R_1 and R_2 are divisorial and let E_1, E_2 be the respective exceptional loci.

The same argument as the one which concludes the proof of theorem 5.3.2 shows that we cannot have $E_1 \cdot V = E_2 \cdot V = 0$, so we can suppose that $E_1 \cdot V > 0$.

If $x \in X$ is a general point then $\mathrm{Locus}(R^1)_{\mathrm{Locus}(V_x)}$ is nonempty and has dimension four, so $E_1 = \mathrm{Locus}(R^1)_{\mathrm{Locus}(V_x)}$; in particular every fiber of R_1 meets $\mathrm{Locus}(V_x)$ and so it is two-dimensional.

Now we apply theorem 3.3.1, we get that $\varphi_1 : X \to Y$ is a blow-down with center a smooth surface S and we repeat the above argument to prove that $Y \simeq \mathbb{P}^5$.

Final step

Let $S \subset \mathbb{P}^5$ be the center of the blow-up, let l be a (bi)secant line of S and let $\tilde{l}$ be the proper transform of l; then

$$-K_X \cdot \tilde{l} = \varphi_1^* \mathcal{O}(6) \cdot l - 2E_1 \cdot \tilde{l} = 2,$$

so the corresponding family on X is unsplit. Since X does not admit unsplit covering families, through the general point of $\mathbb{P}^5$ there is no secant line of S.

It follows that either S is a Veronese surface or S is degenerate.

If S is contained in an hyperplane H and in no three-dimensional linear subspace of $\mathbb{P}^5$, through every point of H there is a secant line, so the strict transform $\widetilde{H}$ of H is the locus of an unsplit family W on X.

Since $\varphi_1^* \mathcal{O}(1) = \varphi_1^* H = \widetilde{H} + kE_1$ with $k > 0$ we have $\widetilde{H} = \varphi_1^* \mathcal{O}(1) - kE_1$, so that $\widetilde{H} \cdot W < 0$. It follows that $\widetilde{H}$ is negative on R_2, so that $\widetilde{H} = E_2$, R_2 is divisorial and $W = R^2$ by lemma 7.1.3.

Again from the canonical bundle formula we have that $E_1 \cdot R^2 = 2$, so $k = 1$ and $E_2 = \widetilde{H} = \varphi_1^* \mathcal{O}(1) - E_1$; in particular $\varphi_1^* \mathcal{O}(1) \cdot R^2 = 1$, and the contraction $\varphi_2 : X \to Z$ is supported by $K_X + 2\varphi_1^* \mathcal{O}(1) + \varphi_2^* A$ for some very ample divisor A on Z. Then φ_2 is equidimensional by [AW98, Corollary 5.8.1], so it is a smooth blow-down.

Computing the canonical bundle of E_2

$$K_{E_2} = -5\varphi_1^*\mathcal{O}(1) + E_1,$$

we find that E_2 is a Fano variety.
E_2 has a $\mathbb{P}^2$-bundle structure over a smooth surface $S' \subset Z$, and since $\rho(E_2) = 2$ we have $S' \simeq \mathbb{P}^2$; by the classification in [SW90] we have that S is a cubic scroll.
Finally, if S is contained in a three-dimensional linear subspace $\Lambda \subset \mathbb{P}^5$ and l is a line in Λ we have

$$-K_X \cdot \tilde{l} = \varphi_1^*\mathcal{O}(6) \cdot \tilde{l} - 2E_1 \cdot \tilde{l} = 6 - 2\deg S,$$

so S has degree ≤ 2 and it cannot be a plane, since the blow-up of $\mathbb{P}^5$ along a two-dimensional plane has a fiber type contraction. $\square$

8

Examples

In this chapter we show the effectiveness of all cases listed in Theorem 3.

$\dim X$	ρ_X	R_1	R_2	R_3	R_4	R_5	
5	2	F	F				a
		F	D_0				b
		F	D_1				c
		F	D_2				d
		F	S				e
		D_2	D_2				f
		D_2	S				g
	3	F	F	F			h
		F	F	S			i
		F	F	D_1			j
		F	F	D_2			k
		F	D_2	D_2			l
	4	F	F	F	F		m
		F	F	F	D_2		n
	5	F	F	F	F	F	o
6	2	F	F				p
		F	D_1				q
		F	D_2				r
		F	S				s
	3	F	F	F			t
7	2	F	F				u
		F	D_2				v
8	2	F	F				w

Observe first of all that cases **o.**, **t.** and **w.** are characterized by Mukai conjecture, and are respectively $(\mathbb{P}^1)^5$, $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^4 \times \mathbb{P}^4$.

8.1 Blow-ups

The first class of examples is given by the blow-ups of $\mathbb{P}^n$ along a linear subspace Λ of codimension $r \geq 2$. The canonical bundle formula

$$K_X = \pi^* K_{\mathbb{P}^n} + (r-1)E$$

yields that X is a Fano variety of pseudoindex

$$i_X = \min\{r-1, n+2-r\};$$

moreover X has a divisorial contraction (the blow-down) and a fiber type contraction associated with the nef bundle $\pi^*(\mathcal{O}_{\mathbb{P}^n}(1)) - E$. What follows is a list of blow-ups of $\mathbb{P}^n$ of pseudoindex $n-3$:

b. $X = Bl_p\mathbb{P}^5$.

d. $X = Bl_{\mathbb{P}^2}\mathbb{P}^5$.

q. $X = Bl_l\mathbb{P}^6$.

r. $X = Bl_{\mathbb{P}^2}\mathbb{P}^6$.

v. $X = Bl_{\mathbb{P}^2}\mathbb{P}^7$.

8.2 Projectivization of vector bundles

Lemma 8.2.1. *Let Y be a Fano variety and let $\mathcal{E}$ be a nef vector bundle of rank r on Y such that $-(K_Y + \det \mathcal{E})$ is ample. Then $X = \mathbb{P}_Y(\mathcal{E} \oplus \mathcal{O}_Y)$ is a Fano variety. Moreover, if we denote by $\widetilde{Y}$ the section corresponding to the surjection $\mathcal{E} \oplus \mathcal{O}_Y \to \mathcal{O}_Y \to 0$ and by R the extremal ray corresponding to the projection map $\pi : X \to Y$, we have that*

$$\mathrm{NE}(X) = \mathrm{NE}(\tilde{Y}) + R.$$

Proof. Let ξ denote the tautological bundle of X; then we can write

$$-K_X = r\xi - \pi^*(K_Y + \det \mathcal{E}).$$

The line bundle $-\pi^*(K_Y + \det \mathcal{E})$ is the pull-back of an ample bundle, so it is nef; moreover it vanishes only on curves contained in the fibers of π, where we know ξ to be positive. Since also ξ is nef, we can conclude that X is a Fano variety.
The tautological bundle ξ vanishes on the special section $\widetilde{Y}$ and, in some cases, on a family of sections of the projection map π; this means that in $\mathrm{NE}(X)$ there exists an extremal face which is isomorphic to $\mathrm{NE}(\widetilde{Y})$, and so to $\mathrm{NE}(Y)$. To conclude the proof, consider the unsplit family V of a minimal curve whose class lies in R and apply remark 5.2.4, noting that in this case $F = \widetilde{Y}$ and so $\mathrm{Locus}(V)_F = X$. □

Remark 8.2.2. If X is the blow-up of $\mathbb{P}^n$ along a linear subspace of codimension $r \geq 2$, then X can be also written as

$$X = \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}(1) \oplus \mathcal{O}^{n-r+1}).$$

The first easy application of lemma 8.2.1, which is in fact a generalization of the construction of blow-ups as described in remark 8.2.2, is the case of a decomposable vector bundle $\mathcal{E} = \oplus \mathcal{O}_{\mathbb{P}^n}(a_i)$, $a_i \geq 0$, over $\mathbb{P}^n$: to guarantee that $-(K_Y + \det \mathcal{E})$ is ample it is enough to assume that $\sum a_i \leq n$, and in this case $X = \mathbb{P}_{\mathbb{P}^n}(\mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^n})$ is a Fano variety of pseudoindex $i_x = \min\{\mathrm{rk}\, \mathcal{E}, n + 1 - \sum a_i\}$.

The section $\widetilde{\mathbb{P}^n}$ which corresponds to the surjection $\mathcal{E} \oplus \mathcal{O} \to \mathcal{O} \to 0$ has the property that each curve contained in $\widetilde{\mathbb{P}^n}$ has trivial intersection with ξ; if besides $a_1 = 0$ then there exists a one-parameter family of sections with this property. The section $\widetilde{\mathbb{P}^n}$, or the locus of the one-parameter family if it exists, are the exceptional locus of the contraction associated with ξ.

c1. $X = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$.

c2. $X = \mathbb{P}_{\mathbb{Q}^3}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$.

e. $X = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$.

s. $X = \mathbb{P}_{\mathbb{P}^4}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$.

The examples that follow are other applications of lemma 8.2.1:

i. Consider the Fano variety

$$Y := \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(1)) = Bl_p\mathbb{P}^4.$$

Lemma 8.2.1 yields that Y has a fiber type contraction on $\mathbb{P}^3$ which comes from its $\mathbb{P}^1$-bundle structure and a divisorial contraction of a $\mathbb{P}^3$ to a point.

Let $\varphi : Y \to \mathbb{P}^3$ be the projection map, and consider on Y the nef line bundle $H = \varphi^*\mathcal{O}_{\mathbb{P}^3}(1)$; then $-(K_Y + H) = 2\xi_Y + 2H$, where ξ_Y denotes the tautological bundle on Y. Since ξ_Y is nef and vanishes only on the special section $\widetilde{\mathbb{P}^3}$, where H is positive, we can apply again lemma 8.2.1 and conclude that

$$X := \mathbb{P}_Y(H \oplus \mathcal{O}_Y)$$

is a Fano variety.

Now let ξ denote the tautological bundle of X and $\pi : X \to Y$ the projection map; then

$$-K_X = 2\xi - \pi^*(K_Y + \det(H \oplus \mathcal{O}_Y)) = 2(\xi + \pi^*(\xi_Y + H)),$$

and since $\pi^*(\xi_Y + H) \cdot f = 0$ and $\xi \cdot f = 1$ for a fiber f of π, we have that $i_X = 2$.

Besides the $\mathbb{P}^1$-bundle contraction $\pi : X \to Y$, X admits a $\mathbb{P}^1$-bundle contraction supported by $\pi^*H + \xi$ and the contraction of a $\mathbb{P}^3 \subset \widetilde{Y}$ to a point.

l1. Here we construct an example of a fivefold where the two divisorial contractions have the same exceptional locus.
Let $Y = \mathbb{P}^2 \times \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)$; since both L and $-(K_Y + L) = \mathcal{O}_Y(2,2)$ are ample, lemma 8.2.1 yields that

$$X = \mathbb{P}_Y(\mathcal{O}_Y \oplus L)$$

is a Fano variety. With the same notation as in the previous example, since $\pi^*L \cdot f = 0$ and $\xi \cdot f = 1$ we can conclude that $i_X = 2$.

In this case X admits a fiber type contraction, which is given by its structure of $\mathbb{P}^1$-bundle, and two divisorial contractions of the special section $\widetilde{Y}$ to $\mathbb{P}^2$, which are in fact smooth blow-downs.

l3. In this example X has two divisorial contractions whose exceptional loci are different but have nonempty intersection.
Let $Y = Bl_l\mathbb{P}^4$; using the same notation as in example **i.** we have that

$$X := \mathbb{P}_Y(H \oplus \mathcal{O}_Y)$$

is a Fano variety of pseudoindex 2.

X admits a $\mathbb{P}^1$-bundle contraction on Y and two divisorial contractions: the first one contracts the special section on X to $\mathbb{P}^2$ (note that $Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(1))$), while the second one contracts the $\mathbb{P}^1$-bundle over the exceptional divisor in $\widetilde{Y}$ to a two-dimensional quadric.

8.3 Products

Finite products of Fano varieties provide a huge class of examples to work with, since the outcome is a Fano variety. In particular we can consider products of the form $Y \times \mathbb{P}^r$: in this case we know that $i_X = \min\{r+1, i_Y\}$ and that X has a natural fiber type contraction given by the projection map $X \to Y$. Moreover, the other contractions of X arise from contractions $\varphi : Y \to Z$ and are given by $\varphi \times id_{\mathbb{P}^r}$.

Examples **a.**, **h.** and **m.** can be realized as the product of $\mathbb{P}^1$ and a Fano fourfold of pseudoindex 2 which admits only fiber type contractions; for a list of these fourfolds see [Wiś90a]. Other examples which can be realized as products are listed below:

j1. $X = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(2)) \times \mathbb{P}^1$.
j2. $X = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(1)) \times \mathbb{P}^1$.
j3. $X = \mathbb{P}_{\mathbb{Q}^3}(\mathcal{O} \oplus \mathcal{O}(1)) \times \mathbb{P}^1$.

k. $X = Bl_l\mathbb{P}^4 \times \mathbb{P}^1$.

n. $X = Bl_p\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$.

p. $X = S \times \mathbb{P}^2$, where S is a del Pezzo fourfold of pseudoindex 3, for example a cubic hypersurface $\subset \mathbb{P}^5$.

u. $X = \mathbb{Q}^4 \times \mathbb{P}^3$.

8.4 Other examples

f1. Let $X = Bl_{S_3}\mathbb{P}^5$, where S_3 is a cubic scroll contained in a hyperplane $H \subset \mathbb{P}^5$; denote by σ the blow-up and by E the exceptional divisor.
Let $\sigma^*\mathcal{O}_{\mathbb{P}^5}(1)$ be the pull-back to X of the hyperplane bundle of $\mathbb{P}^5$, and let $\widetilde{H} = \sigma^*\mathcal{O}(1) - E$ be the strict transform of H; the linear system

$$|L| = \sigma^*|\mathcal{O}(2) - S_3| = |2\sigma^*\mathcal{O}(1) - E|$$

has empty base locus on X and the associated map $\varphi_{|L|}$ gives $\widetilde{H}$ a structure of $\mathbb{P}^2$-bundle over $\mathbb{P}^2$.
Moreover $\widetilde{H}_{|\widetilde{H}} = (L - \sigma^*\mathcal{O}(1))_{|\widetilde{H}}$, so that the restriction of $\widetilde{H}$ to each fiber of $\varphi_{|L|}$ is $\mathcal{O}_{\mathbb{P}^2}(-1)$; we can therefore apply the Nakano contractibility criterion [Nak71], which yields the existence of a manifold $M \supset \mathbb{P}^2$ such that $X \simeq Bl_{\mathbb{P}^2}(M)$ and $\widetilde{H}$ is the exceptional divisor of this blow-up.
Moreover, if we denote by ψ the rational map associated to the linear system $|\mathcal{O}(2) - S_3|$ on $\mathbb{P}^5$ we have that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\varphi_{|L|}"] & \\ \mathbb{P}^5 \arrow[rr, dashed, "\psi"'] & & M \end{tikzcd}$$

One can also prove (see [Fuj81]) that M is isomorphic to the hyperplane section associated with the Plücker embedding of the Grassmannian $G(1,4) \subset \mathbb{P}^9$, so (see [Fuj81, 7.1]) M is a Del Pezzo variety.
Note that since $\rho_X = 2$ and X has two smooth blow-downs, $-K_X$ is positive on the entire cone $\mathrm{NE}(X)$, so X is a Fano variety. Moreover, we can write

$$-K_X = 6\sigma^*\mathcal{O}(1) - 2E = 2(3\sigma^*\mathcal{O}(1) - E),$$

so X has pseudoindex 2.

f2. Let $X = Bl_V\mathbb{P}^5$, where V is a Veronese surface. Denote by σ the blow-up and by E the exceptional divisor.
Consider on $\mathbb{P}^5$ the linear system $|\mathcal{O}_{\mathbb{P}^5}(2) - V|$ of the quadrics containing V, and denote by $F : \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$ the associated rational map; call V' the exceptional locus of F^{-1} and let $\sigma' : X' \to \mathbb{P}^5$ be the blow-up of $\mathbb{P}^5$ along V'.

$$\begin{tikzcd} X \arrow[r, phantom, "\cong"] \arrow[d, "\sigma"'] & X' \arrow[d, "\sigma'"] \\ \mathbb{P}^5 \arrow[r, dashed, "F"'] & \mathbb{P}^5 \end{tikzcd}$$

One can prove (see [ESB89, 2.0.2]) that $X' \simeq X$, that the exceptional divisors of the two blow-ups satisfy the relations
$E = 2\sigma^*\mathcal{O}(1) - \sigma'^*\mathcal{O}(1)$
$E' = 2\sigma'^*\mathcal{O}(1) - \sigma^*\mathcal{O}(1)$,

and that the map F is an involution [ESB89, Theorem 2.6].
As in the previous example, since $\rho_X = 2$ and X has two smooth blow-downs, $-K_X$ is positive on the entire cone $\mathrm{NE}(X)$, so X is a Fano variety, and from the canonical bundle formula

$$-K_X = 6\sigma^*\mathcal{O}_{\mathbb{P}^5}(1) - 2E = 2(3\sigma^*\mathcal{O}_{\mathbb{P}^5}(1) - E),$$

we have that X has pseudoindex 2.

g. Let $X = Bl_{\mathbb{Q}^2}\mathbb{P}^5$, where $\mathbb{Q}^2 \subset \mathbb{P}^5$ is a smooth two-dimensional quadric, denote by σ the blow-up and by E the exceptional divisor; then

$$-K_X = 6\sigma^*\mathcal{O}_{\mathbb{P}^5}(1) - 2E.$$

For every curve $C \subset X$ which is not contained in E, we have that $\sigma(C)$ is a curve in $\mathbb{P}^5$ of a certain degree d, and the sum of the multiplicities of the points of intersection of $\sigma(C)$ and $\mathbb{Q}^2$ is $\leq 2d$. This implies that

$$-K_X \cdot C \geq 6d - 4d \geq 2d.$$

The exceptional divisor E can be written as

$$E = \mathbb{P}(\mathcal{N}^*_{\mathbb{Q}^2|\mathbb{P}^5}) = \mathbb{P}(\mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)),$$

and $E_{|E} = -\xi_{\mathcal{N}^*}$. If we denote by Q the section of $\sigma : E \to \mathbb{Q}^2$ which corresponds to $\mathcal{O}(-2)$, then $\mathrm{NE}(E) = \langle [l_1], [l_2], [l_3] \rangle$, where l_1 and l_2 correspond to the two rulings of Q and l_3 is a line in a fiber of σ.
If we write K_E as $-3\xi - 6\sigma^*\mathcal{O}_{\mathbb{Q}^2}(1)$, the adjunction formula yields

$$-K_{X_{|E}} = -K_E + E_{|E} = 2\xi + 6\sigma^*\mathcal{O}(1),$$

so $-K_X \cdot l_i = 2$ for every i, hence X is a Fano variety of pseudoindex 2.

The line bundle $2\sigma^*\mathcal{O}(1) - E$ is nef on X, and it vanishes on the strict transform of the $\mathbb{P}^3 \subset \mathbb{P}^5$ which contains $\mathbb{Q}^2$; hence it is the supporting divisor of the small contraction of $\mathbb{P}^3$ to a point.

12. The blow-up of $\mathbb{P}^5$ along two disjoint $\mathbb{P}^2$s provides an example of a fivefold with two divisorial contractions with disjoint exceptional loci.

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